Hypercyclic algebras: what we do and do not know

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For the conference **Frontiers of Operator Theory** at CIRM, Marseille, France, from november 29th to december 3rd 2021

What is it?

When $T : X \to X$ is a hypercyclic operator acting on a Fréchet space X which has the structure of algebra, then we may look for a subalgebra A of X such that all non-zero vectors are hypercyclic for T. Such a structure is what we call a **hypercyclic algebra** (we will use the abbreviation H.A.)

Where did this come from?

Spaceability is a well studied subject in hypercyclicity under the name of **hypercyclic subspaces**. When the underlying space is an algebra, it is natural to study algebrability in the same context. Whereas Aron, Conejero, Peris and Seoane-Sepúlveda [1] have shown that no translation on $H(\mathbb{C})$ has a H.A., Bayart & Matheron [4] and Shkarin [10] have found such a structure for $D : f \mapsto f'$ on $H(\mathbb{C})$. Since then the theory has significantly evolved, although many problems are yet to be settled.

Hypercyclic algebras

We are in the know!

- No translation $\tau_a : f \mapsto f(\cdot + a)$ on $H(\mathbb{C})$ has a H.A. [1].
- However, translations $\tau_a : C^\infty(\mathbb{R}, \mathbb{C}) \to C^\infty(\mathbb{R}, \mathbb{C})$ have H.A. for all $a \in \mathbb{R}^*$ (see [7]).
- No comp. operator $C_\phi : f \mapsto f \circ \phi$ on $H(\Omega)$ has a H.A. [7].
- However, $P(C_\phi)$ can have H.A. (see [2]).
- Convolution operators $\phi(D)$ on $H(\mathbb{C})$ with its canonical product have H.A. under conditions on $\phi \in H(\mathbb{C})$ [2,3,5,7,8].
- Operators $P(B)$ on $\ell_1(\mathbb{N})$ with the convolution product have H.A. under conditions on $P \in \mathbb{C}[X]$ (see [2]).
- Weighted backward shifts B_w on a Fréchet sequence algebra X have H.A. with the coordinatewise and convolution products under conditions on w or on X (see [3,9]).

But we don't know...

- If $\phi(D)$ for $|\phi(0)| > 1$ can have a H.A. in general (for example $2 + D$).
- If left multiplication operators $L_T : S \mapsto T \circ S$ on $\mathcal{K}(H)$ with SOT can have H.A.
- Which $P(B_w)$ on $\ell_1(\mathbb{N})$ with the Cauchy product has H.A.
- $\phi(D)$ or $P(B)$ as before but for the coordinatewise product.
- Which composition operators $C_\phi : C^\infty(\Omega, \mathbb{C}) \to C^\infty(\Omega, \mathbb{C})$ has a H.A. when $\Omega \subset \mathbb{R}^d$, $d > 1$.

Closed hypercyclic algebras

As $HC(T) \cup \{0\}$ is always dense lineable (Herrero-Bourdon theorem), the interesting results come from the concept of spaceability. From this notion derives that of hypercyclic subspaces: they are closed infinite-dimensional subspaces of $HC(T) \cup \{0\}$. For algebrability, however, the question is fundamentally difficult. Even harder is the search for **closed hypercyclic algebras**.

We are in the know!

- Any non-trivial translation operator on $C^\infty(\mathbb{R}, \mathbb{C})$ have a closed H.A. [K. Grosse-Erdmann and D. Papathanasiou].
- No backward shift with the coordinatewise product have a closed H.A. (see [3]).
- No convolution operator on $H(\mathbb{C})$ induced by a polynomial have a closed H.A. (see [3]).

But we don't know...

- Is there a backward shift with a closed hypercyclic algebra (even on ω) for the Cauchy product?
- Could a non-polynomial entire function induce a convolution operator with a hypercyclic algebra?

More related topics

- Common and disjoint hypercyclic algebras
- Frequently and upper frequently hypercyclic algebras
- Γ-supercyclic algebras

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