The origins of Linear Dynamics

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Surface transformations and their dynamical applications Article by G. Birkhoff (1920) - Page 111

Chapter VI. The general point group.

§ 61. Classification of transformations T.

Before entering upon further discussion of the behavior of points under T, we shall effect a classification which is fundamental.

A transformation T will be called *transitive* if, for any pair of points P and Q on S nearby points P' and Q' respectively can be found such that $Q' = T_n(P')$.

A transformation T is *intransitive* in the contrary case.

It seems highly probable that the transitive case is to be regarded as the general case.

§ 62. The transitive case.

We commence with the transitive case.

In the transitive case all of the recurrent point groups are unstable.[1]

In fact it has been observed earlier that a stable recurrent point group leads to continua forming part of S, which are invariant as a set under T and lie near the point group. Hence if we take a point P outside of these continua

Surface transformations and their dynamical applications
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Suppose now that we designate any point whose α or ω limit points do not form all of S as a *special* point. All of the points belonging to recurrent point groups or points asymptotic to such point groups are of this type.

Points which are not special evidently pass into the neighborhood of all points of S' under iteration of T or T_{-1}. Such points we term *general*.

In the transitive case the general points are everywhere dense in S.

To see this we divide S into a large number of regions S' of small diameter d, and consider the set of points P whose iterates do not enter within all of the regions S'. Such points P evidently form a closed set of points, M say.

This set M is nowhere dense in S. In the contrary case suppose M to fill a small region σ'. Now there are only a finite set of regions S' and thus only a finite number of combinations of less than all of them. Divide the points of σ' into the finite number of closed sets according to the regions S' which the points enter. Thus σ' is divided into a finite number of closed sets, at least one of which therefore fills some neighborhood σ'' of σ' densely. We recall that a finite or denumerably infinite set of nowhere dense closed sets cannot fill a complete neighborhood. But the existence of such a region σ'' contradicts the condition that T is transitive. Thus M is nowhere dense.

Again choose a set of subregions S'' of the regions S' of diameter less than

Quelques théorèmes sur le mouvement des systèmes dynamiques - Article by Birkhoff (1912) - Page 311

III. — Mouvement récurrent.

On vient de voir que tout mouvement M, positivement ou négativement stable, possède un ensemble *fermé* de mouvements limites M′, positivement et négativement stables. Il s'ensuit immédiatement que l'ensemble des mouvements limites alpha et oméga de tout mouvement de M′, ou bien coïncide avec M′, ou bien est un sous-ensemble de M′ [1].

Définition. — Tout ensemble fermé M′ de mouvements positivement et négativement stables, tel que tout mouvement de M′ admet M′ pour son ensemble de mouvements limites alpha, ainsi que pour son ensemble de mouvements limites oméga, sera appelé un ensemble *minimal;* et tout mouvement de M′ sera appelé un mouvement *récurrent.*

Par définition, un mouvement récurrent M est stable. De plus, tout point de la courbe représentative est à la fois un point-limite

Invariant subspace problem : A. Beurling (1949) and von Neumann?

Context: X is an infinite dimensional topological vector space and $\mathcal{L}(X)$ denotes the algebra of bounded operators over X. An **invariant subspace** F of T is a closed subspace of $F \subset X$ such that $T(F) \subset F$. An **invariant subset** A of T is a closed subset $A \subset X$ such that $T(A) \subset A$.

Does every $T \in \mathcal{L}(X)$ have a non-trivial invariant subspace?

- X is Banach: No (P. Enflo (1976), C. Read on ℓ_1 (1985)).
- X is Hilbert non-separable: Yes.
- X is Hilbert: **Open**.

Remark: an operator T lacks invariant subspaces iff every $x \in X \setminus \{0\}$ is *cyclic* for T, that is, $\operatorname{span}(\operatorname{orb}(x, T))$ is dense in X.

Does every $T \in \mathcal{L}(X)$ have a non-trivial invariant **subset***?*

- X is Banach: No (C. Read on ℓ_1 (1988)).
- X is any Hilbert : **Open**.

Remark: an operator T lacks invariant subsets iff the dynamical system $(X \setminus \{0\}, T)$ is *minimal* in the sense of Birkhoff.

Démonstration d'un théorème élémentaire sur les fonctions entières - Note by G. Birkhoff (1929)

THÉORIE DES FONCTIONS. — *Démonstration d'un théorème élémentaire sur les fonctions entières.* Note (1) de M. **George D. Birkhoff.**

Soit $f(x)$ une fonction entière de la variable x. Nous dirons que la fonction entière $g(x)$ est une fonction limite de $f(x)$, si l'on peut écrire

$$g(x) = \lim_{n=\infty} f(x + c_n),$$

où $c_1, c_2, \ldots$ sont des constantes convenablement choisies, et où la convergence doit être uniforme dans toute région bornée du plan de la variable x. La théorie des fonctions limites ne semble pas avoir été développée.

Le but de notre Note est de démontrer le théorème élémentaire qui suit :

Il existe des fonctions entières dont toutes les fonctions entières sont des fonctions limites.

Statement and proof of Birkhoff's theorem - Part 1/5

Birkhoff's Theorem (1929)

There is an entire function $f \in H(\mathbb{C})$ such that any other $g \in H(\mathbb{C})$ is a limit function of f, that is, there is $(a_n)_n$ such that

$$g(x) = \lim_{n\to\infty} f(x + a_n)$$

uniformly on compact subsets of $\mathbb{C}$.

Proof: Let $(p_n)_n$ be a countable family of polynomials which is dense in $H(\mathbb{C})$. It is clear that one only needs to construct $f \in H(\mathbb{C})$ such that any polynomial in the family $(p_n)_n$ is one of its limit functions.

Statement and proof of Birkhoff's theorem - Part 2/5

We start by fixing $\varepsilon_n \downarrow 0$ in $\mathbb{R}_+$. Our function f will take the form

$$f(x) = u_1(x) + u_2(x) + \cdots ,$$

with

$$u_n(x) = p_n(x - a_n)e^{-c_n(x-a_n)^2},$$

where the sequences $(c_n)_n, (a_n)_n$ as well as the closed disks $(C_n)_n$ centered in a_n with radius n are to be defined satisfying:

(1) $|u_n(x) - p_n(x - a_n)| \leq \dfrac{\varepsilon_n}{2^n}$ for all $x \in C_n$;

(2) $|u_i(x)| \leq \dfrac{\varepsilon_n}{2^i}, i = 1, ..., n-1$, for all $x \in C_n$;

(3) $|u_n(x)| \leq \dfrac{\varepsilon_i}{2^n}$, for all $x \in C_i, i = 1, ..., n-1$;

(4) $|u_n(x)| \leq \dfrac{1}{2^n}$ for all $|x| \leq n$.

Statement and proof of Birkhoff's theorem - Part 3/5

The conclusion is that, for any polynomial p_n in the initial dense family and all $|x| \leq n$, we have $x + a_n \in C_n$ and

$$
\begin{aligned}
|f(x+a_n) - p_n(x)| &= \left| \sum_{i=1}^{\infty} u_i(x+a_n) - p_n(x) \right| \\
&\leq |u_n(x+a_n) - p_n(x)| + \sum_{i=1}^{n-1} |u_i(x+a_n)| + \sum_{i>n} |u_i(x+a_n)| \\
&\leq \frac{\varepsilon_n}{2^n} + \sum_{i=1}^{n-1} \frac{\varepsilon_n}{2^i} + \sum_{i>n} \frac{\varepsilon_n}{2^i} = \varepsilon_n.
\end{aligned}
$$

In other words,

$$|x| \leq n \Longrightarrow |f(x+a_n) - p_n(x)| \leq \varepsilon_n,$$

hence p_n is a limit function of f.

Statement and proof of Birkhoff's theorem - Part 4/5

For the construction of $(c_n)_n$ and $(a_n)_n$, we first begin by choosing c_1 sufficiently small so that

$$|z| \leq 1 \Longrightarrow |p_1(z)e^{-c_1 z^2} - p_1(z)| \leq \frac{\varepsilon_1}{2}.$$

This implies condition (1) for any choice of a_1. Then, we choose a_1 big enough so that

$$|x| \leq 1 \Longrightarrow |u_1(x)| = |p_1(x - a_1)e^{-c_1(x-a_1)^2}| \leq \frac{1}{2}.$$

With this we verify conditions (1) and (4) at rank n. Conditions (2) and (3) are empty.

Statement and proof of Birkhoff's theorem - Part 5/5

Suppose that we have constructed $c_1, \ldots, c_{n-1}$ and $a_1, \ldots, a_{n-1}$ satisfying (1)-(4). Then we take c_n small enough so that

$$|z| \leq n \Longrightarrow |p_n(z)e^{-c_n z^2} - p_n(z)| \leq \frac{\varepsilon_n}{2^n}.$$

This implies (1) for any choice of a_n. Then, we choose a_n big enough so that

$$x \in C_n \Longrightarrow |u_i(x)| = |p_i(x - a_i)e^{-c_n(x-a_i)^2}| \leq \frac{\varepsilon_n}{2^i}, i = 1, \ldots, n-1,$$

$$x \in C_i \Longrightarrow |u_n(x)| = |p_n(x - a_n)e^{-c_n(x-a_n)^2}| \leq \frac{\varepsilon_i}{2^n}, i = 1, \ldots, n-1$$

$$|x| \leq n \Longrightarrow |u_n(x)| = |p_n(x - a_n)e^{-c_n(x-a_n)^2}| \leq \frac{1}{2^n},$$

what implies (2)-(4) at rank n and completes the proof. ■

Modern statement of Birkhoff's theorem

For any $a \in \mathbb{C}$, we define the (so called Birkhoff) operator of translation by a as

$$\begin{array}{rccl} \tau_a : & H(\mathbb{C}) & \longrightarrow & H(\mathbb{C}) \\ & f & \longmapsto & \tau_a(f) : x \in \mathbb{C} \mapsto f(x+a) \end{array}$$

A family of operators $(T_a)_{a\in\Gamma}$ acting on a Fréchet space X is said to be **universal** when there exists $f \in X$ such that $\overline{\{T_a(f) : a \in \Gamma\}} = X$.

Bihkroff's theorem

The family of all translations $(\tau_a)_{a\in\mathbb{C}}$ is universal.

Bihkroff's theorem generalization (proof corollary)

For any $a \in \mathbb{C}\backslash\{0\}$, the linear operator τ_a has a dense orbit.

Sequences of derivatives and normal families
Article by MacLane (1952) - Page 84

3. An Ubiquitous Entire Function.

Theorem 6. *There exists an entire function $U(z)$ satisfying:*

1^0. $|U(z)| = O(e^{(1+\varepsilon)|z|})$ *for any* $\varepsilon > 0$, *and*

2^0. *If D is any simply connected domain in the z-plane and* $\varphi(z)$ *is any function holomorphic in D, then there exists a subsequence,* n_j, $1 \leqq j < \infty$, *of the positive integers such that*

$$U^{(n_j)}(z) \to \varphi(z), \qquad j \to \infty, \tag{40}$$

uniformly in any compact subset of D.

Proof. Let $P_n(z)$, $n = 1, 2, \ldots$, be any sequence of polynomials with the property: for any D and $\varphi(z)$ as in 2^0, there exists n_j such that

$$P_{n_j}(z) \to \varphi(z) \tag{41}$$

MacLane's theorem (1952)

The linear operator $D : f \longmapsto f'$ of complex derivation on $H(\mathbb{C})$ has a dense orbit.

Topological Dynamics
Book by Gottshalk and Hedlund (1955)

9.02. **Definition.** Let $x \in X$. The transformation group (X, T) is said to be *transitive at* x and the point x is said to be *transitive under* (X, T) provided that if U is a nonvacuous open subset of X, then there exists $t \in T$ such that $xt \in U$.

The transformation group (X, T) is said to be {*pointwise*}{*point*} *transitive* provided that (X, T) is transitive at {every}{some} point of X.

The transformation group (X, T) is said to be (*regionally*) *transitive* provided that if U and V are nonvacuous open subsets of X, then there exists $t \in T$ such that $Ut \cap V \neq \emptyset$.

9.20. **Theorem.** *Let X be a complete separable metric space. Then the following statements are pairwise equivalent:*

(1) *(X, T) is transitive.*

(2) *(X, T) is point transitive.*

(3) *The set of all transitive points is an invariant residual G_δ subset of X.*

On orbits of vectors
Article by Rolewicz (1969)

THEOREM 1. *Let X be either $l^p (1 \leqslant p < +\infty)$ or c_0. For any arbitrary real $a > 1$, there are a linear continuos operator A and an element x_0 such that the orbit $\mathcal{O}(A : x_0)$ is dense in the whole of X.*

Proof. Let S be the left shift operator

$$S(\{x_1, x_2, x_3, \ldots\ldots\}) = \{x_2, x_3, \ldots\ldots\}$$

and S' be the right shift operator

$$S'(\{x_1, x_2, x_3, \ldots\}) = \{0, x_1, x_2, \ldots\}.$$

Let $\boxed{A = aS}$ and $B = S'/a$. Then $\|A\| = a$, $\|B\| = 1/a$, $AB = I$, where I, as usual, denotes the identity operator.

Now we shall start to construct the vector x_0. Let x^n be a sequence

Rolewicz's theorem (1969)

For all $\lambda > 1$, the linear operator $\lambda B : (x_n)_n \mapsto (\lambda x_{n+1})_n$, acting on $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N}), 1 \leq p < \infty$, admit a dense orbit.

Invariant closed sets for linear operators
Ph.D thesis of Kitai (1982) - unpublished

Kitai's thesis marks the beginning of the Theory of Hypercyclicity. It includes:

- clarification/establishment of basic concepts;
- plenty of examples/countexeamples were given;
- development of techniques that have been used since then;

Kitai's criterion (1982)

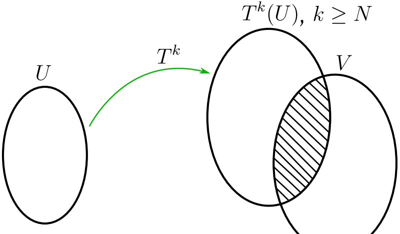
Let X be a separable Fréchet space and $T \in \mathcal{L}(X)$. Suppose that there are dense subsets $X_0, Y_0 \subset X$ and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0, y \in Y_0$,

(i) $T^n x \to 0$,

(ii) $S^n y \to 0$,

(iii) $TSy = y$.

Then T is mixing.



The term "hypercyclicity"

- A vector with a dense orbit was called "orbital" by Kitai and "universal" by Gethner and Shapiro.
- similar notions are those of cyclic and supercyclic vectors. The latter was introduced in 1974 by H. M. Hilden and L. J. Wallen. The notion of cyclic vector seems to be much older.

In 1984, B. Beauzamy introduces the term "hypercyclicity". He finds an uncountable dense set of hypercyclic vectors for an operator on a Banach space. He insists in the idea that Rolewicz has found an operator that has *one* hypercyclic vector. His findings were published in:

B. Beauzamy. **Un opérateur, sur un espace de Banach, avec un ensemble non-dénombrable, dense, de vecteurs hypercycliques**. In Séminaire de géométrie des espaces de Banach, volume 18 of Publ. Math. Univ. Paris VII, pp. 149–175. Universite Paris VII, 1984.

Universal vectors for Operators on Spaces of Holomorphic Functions - Article by Gethner and Shapiro (1987)

- Independently of Kitai, Gethner and Shapiro show in 1987 the Hypercyclicity Criterion using a Baire argument.
- They obtain applications to Birkhoff, MacLane and Rolewicz operators (and more general backward shifts).
- Their method produces G_δ-dense sets of universal vectors.

Operators with Dense, Invariant Cyclic Manifolds. Article by Godefroy and Shapiro (1991)

In 1991, Godefroy and Shapiro organize/establish the bases of the theory of Linear Dynamical Systems.

- They adopt Devaney's chaos for the linear setting. we say that a system (X, T) is **chaotic** when:
  - T is hypercyclic,
  - T has a dense set of periodic points,
  - T has sensitive dependence on initial conditions.

  Obs.: They have shown that sensitive dependence on initial conditions for linear dynamical systems in implied by hypercyclicity $+$ linearity.
- They finally link, once and for all, topological transitivity to hypercyclicity for linear dynamical systems.
- They unify Birkhoff and MacLane's theorems by showing that any operator that commutes with all translations are hypercyclic as long as they are not a scalar multiple of the identity.
- And much more.

What should I read?

- Survey: K.-G. Grosse-Erdmann (1999). **Universal Families and Hypercyclic Vectors**.
- Notes: J. Shapiro (2001). **Notes on the Dynamics of Linear Operators**.
- Book 1: F. Bayart and É. Matheron (2009). **Dynamics of Linear Operators**.
- Book 2: K.-G. Grosse-Erdmann and A. Peris (2011). **Linear Chaos**.
- Survey: C. Gilmore (2020). **Linear Dynamical Systems**.

Why is transitivity so important?

Short answer: **versatility!**

Long answer: **a lot of versatility!**

Theorem (Modern Birkhoff's transitivity theorem)

Let X be a separable F-space space and $T \in \mathcal{L}(X)$. Suppose that, for all $U, V \subset X$ open and non-empty, there are $u \in U$ and $N \in \mathbb{N}$ such that $T^N(u) \in V$. Then T has a G_δ-dense set of hypercyclic vectors.

One can find adaptations of this result giving G_δ-dense sets of vectors with many properties linked to hypercyclicity. Let us name some of them.

- Common hypercyclicity: elements of proof/statement in a paper by Costakis and Sambarino (2004).
- Disjoint hypercyclicity: Bès and Peris (2007).
- Hypercyclic algebras: Bayart and Matheron (2009), later improved by Bayart, CJr, Papathanasiou (2021).
- $\mathcal{U}$-frequent hypercyclicity: Grosse-Erdmann and Bonilla (2018).
- $\mathcal{U}$-frequent hypercyclic algebras: Bayart, CJr, Papathanasiou (2021).
- Common and disjoint hypercyclic algebras: Bayart, CJr, Papathanasiou (*to appear*).

Transitivity is (often) so easy to apply!

Theorem (Bihkroff's first result)

For any $a \in \mathbb{C}\backslash\{0\}$, the translation $\tau_a : H(\mathbb{C}) \to H(\mathbb{C})$ is hypercyclic.

Proof:

Some transitivity-like criteria

Theorem (Criterion for common hypercyclicity)

Let $(T_\lambda)_{\lambda\in\Gamma}$ be a continuous family of operators acting on the same separable F-space X, with Γ σ-compact. Suppose that, for all $K\subset\Gamma$ compact and all $U,V\subset X$ open and non-empty, there is $u\in U$ such that, for all $\lambda\in K$, one can find $N\in\mathbb{N}$ with $T_\lambda^N(u)\in V$. Then $(T_\lambda)_{\lambda\in\Gamma}$ admits a G_δ-dense set of common hypercyclic vectors.

Some transitivity-like criteria

Theorem (Criterion for common hypercyclicity)

Let $(T_\lambda)_{\lambda\in\Gamma}$ be a continuous family of operators acting on the same separable F-space X, with Γ σ-compact. Suppose that, for all $K\subset\Gamma$ compact and all $U,V\subset X$ open and non-empty, there is $u\in U$ such that, for all $\lambda\in K$, one can find $N\in\mathbb{N}$ with $T_\lambda^N(u)\in V$. Then $(T_\lambda)_{\lambda\in\Gamma}$ admits a G_δ-dense set of common hypercyclic vectors.

Theorem (Criterion for disjoint hypercyclicity)

Let X be an F-algebra and let $T_1,T_2\in\mathcal{L}(X)$. Suppose that for all $U,V_1,V_2\subset X$ open and non-empty, there is $u\in U$ and $N\in\mathbb{N}$ such that $T_1^N u\in V_1$, $T_2^N u\in V_2$. Then there is a G_δ-dense set of vectors $u\in X$ such that $\{(T_1^n u, T_2^n u) : n\in\mathbb{N}\}$ is dense in $X\times X$.

Some transitivity-like criteria

Theorem (Criterion for hypercyclic algebras)

Let X be an F-algebra and let $T \in \mathcal{L}(X)$. Assume that, for all $I \in \mathcal{P}_f(\mathbb{N})\backslash\{\varnothing\}$, there is $m_0 \in I$ such that, for all $U, V \subset X$ open and non-empty and all neighborhood W of 0, one can find $u \in U$ and $N \in \mathbb{N}$ such that

$$\begin{cases} T^N u^n \in W \text{ for } n \in I\backslash\{m_0\}, \\ T^N u^{m_0} \in V. \end{cases}$$

Then there is a G_δ-dense subset of X generating a hypercyclic algebra for T.

Some transitivity-like criteria

Theorem (Criterion for $\mathcal{U}$-frequent hypercyclicity)

Let X be a separable Fréchet space and $T \in \mathcal{L}(X)$. Suppose that, for all $V \subset X$ open and non-empty, there exists $\delta > 0$ such that, for all $U \subset X$ open and non-empty, for all $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ such that

$$\frac{\#\{p \leq N : T^p u \in V\}}{N+1} > \delta.$$

Then T has a G_δ-dense set of $\mathcal{U}$-frequent hypercyclic vectors.

Theorem (Criterion for $\mathcal{U}$-frequent hypercyclic algebras)

Let X be a separable Fréchet space and $T \in \mathcal{L}(X)$. Suppose that for all $I \in \mathcal{P}_f(\mathbb{N})\backslash\{\varnothing\}$, there is $m_0 \in I$ satisfying the following: for all $V \subset X$ open and non-empty and all neighborhood W of 0, there exists $\delta > 0$ such that, for all $U \subset X$ open and non-empty and all $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ satisfying

$$\frac{\#\{p \leq N : T^p u^n \in W,\ \text{for } n \in I\backslash\{m_0\}, \text{ and } T^p u^{m_0} \in V\}}{N+1} > \delta.$$

Then there is a G_δ-dense set of vectors generating a $\mathcal{U}$FHC algebra for T.

Some transitivity-like criteria

Theorem (Criterion for disjoint hypercyclic algebras)

Let X be an F-algebra and let $T_1, T_2 \in \mathcal{L}(X)$. Suppose that, for all $I \subset \mathcal{P}_f(\mathbb{N})\backslash\{\varnothing\}$, there is $m_0 \in I$ such that, for all $U, V_1, V_2 \subset X$ open and non-empty and all neighborhood W of 0, there is $u \in U$ and $N \in \mathbb{N}$ such that

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W, \text{ for all } n \in I, n \neq m_0, \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

Then there is a G_δ-dense set of vectors generating a disjoint hypercyclic algebra for (T_1, T_2).

Some transitivity-like criteria

Theorem (Criterion for common hypercyclic algebras)

Let $(T_\lambda)_{\lambda\in\Gamma}$ be a continuous family of operators acting on the same separable F-space X, with Γ σ-compact. Suppose that, for all $K \subset \Gamma$ compact and all $I \subset \mathcal{P}_f(\mathbb{N})\backslash\{\varnothing\}$, there is $m_0 \in I$ such that, for all $U, V \subset X$ open and non-empty and all neighborhood W of 0, one can find $u \in U$ such that, for all $\lambda \in K$, there is $N \in \mathbb{N}$ satisfying

$$\begin{cases} T_\lambda^N u^n \in W \text{ for } m \in I\backslash\{m_0\}, \\ T_\lambda^N u^{m_0} \in V. \end{cases}$$

Then there is a G_δ-dense set of vectors generating a common hypercyclic algebra for $(T_\lambda)_{\lambda\in\Lambda}$.

First common hypercyclicity results

Theorem (Abakoumov and Gordon (2003))

For $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N}), 1 \leq p < \infty$, the family $(\lambda B)_{\lambda > 1}$ has a common hypercyclic vector. (Even a G_δ-dense set of them, actually!)

Borichev's example

For any $\Lambda \subset (1, +\infty) \times (1, +\infty)$ with positive Lebesgue measure, $(\lambda B \times \mu B)_{(\lambda,\mu)\in\Lambda}$ has no common hypercyclic vector on $\ell_1(\mathbb{N}) \times \ell_1(\mathbb{N})$.

Theorem (Bayart, CJr, Menet (2021))

Let $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N}), 1 \leq p < \infty$. If $\Lambda \subset (1, +\infty)^d, d \geq 1$, is a Lipschitz curve, then $(\lambda_1 B \times \cdots \times \lambda_d B)_{\lambda\in\Lambda}$ has a common hypercyclic vector on X^d.

Some results in multiple dimensions

Theorem (Bayart, CJr, Menet (2021))

(i) *Let* $X = c_0(\mathbb{N})$ *or* $\ell_p(\mathbb{N})$ *(*$1 \leq p < \infty$*), let* $\alpha \in (0,1]$ *and let* $(w(a))_{a>0}$ *be given by* $w_n(a) = 1 + \frac{\lambda}{n^{1-\alpha}}$ *for all* $n \geq 1$*. Suppose that, for* $\Lambda \subset (0,+\infty)^d$*, the family* $(B_{w(\lambda_1)} \times \overset{d}{\cdots} \times B_{w(\lambda_d)})_{\lambda \in \Lambda}$ *has a common hypercyclic vector. Then* $\dim_{\mathcal{H}}(\Lambda) \leq 1/\alpha$*.*

(ii) *Let* X *and* $(w(a))_{a>0}$ *as before. If* $\alpha < 1/d$*, then the family*

$$(B_{w(\lambda_1)} \times \overset{d}{\cdots} \times B_{w(\lambda_d)})_{\lambda \in (0,+\infty)^d}$$

has a common hypercyclic vector.

Some results in multiple dimensions

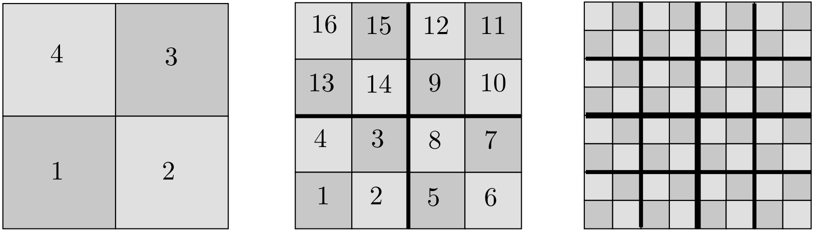
Some ideas :

$$\begin{aligned}
\sum_{j\neq i} B^{n_i}_{w(x)} F^{n_j}_{w(x_j)^{-1}}(e_0) &= \sum_{j=i+1}^{q} B^{n_i}_{w(x)} F^{n_j}_{w(x_j)^{-1}}(e_0) \\
&= \sum_{j=i+1}^{q} \frac{w_{n_j-n_i+1}(x)\cdots w_{n_j}(x)}{w_1(x_j)\cdots w_{n_j}(x_j)} e_{n_j-n_i} \\
&= \sum_{j=i+1}^{q} \frac{\exp(x n_j^\alpha - x(n_j-n_i)^\alpha)}{\exp(x_j n_j^\alpha)} e_{n_j-n_i} \\
&= \sum_{j=i+1}^{q} \frac{1}{\exp((x_j-x)n_j^\alpha + x(n_j-n_i)^\alpha)} e_{n_j-n_i}
\end{aligned}$$

So we want, for all $j > i$,

$$(x_j - x_i)n_j^\alpha + x_i(n_j-n_i)^\alpha \gg 0 \quad \text{and} \quad (y_j - y_i)n_j^\alpha + y_i(n_j-n_i)^\alpha \gg 0.$$

Some results in multiple dimensions



But we need $\alpha < \frac{1}{2}$ for the construction of the associated sequence.
What we know for $w_n(x) = 1 + \frac{x}{n^{1-\alpha}}$:

$$\alpha < 1/2 \quad \Longrightarrow \quad \big(B_{w(x)} \times B_{w(y)}\big)_{x,y>0} \text{ is common hypercyclic;}$$

$$\alpha > 1/2 \quad \Longrightarrow \quad \big(B_{w(x)} \times B_{w(y)}\big)_{x,y>0} \text{ is not common hypercyclic;.}$$

Open question: what can we say when $\alpha = 1/2$?

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