Fractals and common hypercyclicity

Advanced Courses in Operator Theory and Complex Analysis

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Hypercyclicity and common hypercyclicity

- Let X be a separable Banach space and $T : X \to X$ be linear and continuous (= operator).

We say that $x \in X$ is a **hypercyclic vector** for T when its orbit $Orb(x, T) := \{T^n x : n \geq 0\}$ is dense in X. If such a vector exists we say that T is a **hypercyclic operator** on X. The set of hypercyclic vectors for an operator T is denoted by $HC(T)$.

- Let $d \geq 1$, let $\Lambda \subset \mathbb{R}^d$ be compact and let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators acting on X such that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous (= continuous family).

We say that $x \in X$ is a **common hypercyclic vector** for $(T_\lambda)_{\lambda \in \Lambda}$ when $x \in HC(T_\lambda)$ for all $\lambda \in \Lambda$. If such a vector exists we say that $(T_\lambda)_{\lambda \in \Lambda}$ is a **common hypercyclic family** on X.

- Gilles Godefroy and Joel Shapiro (1991), Héctor Salas (1999)
- Evgeny Abakumov and Julia Gordon (2003), Alfred Peris (2000/2001)
- Frédéric Bayart (2004)
- George Costakis and Martín Sambarino (2004)
- Simultaneity of many dynamical properties are studied since 2004
- Frédéric Bayart, CJr, Quentin Menet (2022)

Context: continuous families $(T_\lambda)_{\lambda\in\Lambda}$, with Λ compact, acting on the same space X in which there is $\mathcal{D}\subset X$ dense such that T_λ has a right inverse $S_\lambda : \mathcal{D}\to X$, for all $\lambda\in\Lambda$.

Very first criterion for general families of operators

Theorem (Costakis and Sambarino (2004))

Assume that, for each $v \in \mathcal{D}$ *and each compact interval* K *inside* $\Lambda \subset \mathbb{R}$*, the following properties hold true, where* $\lambda, \mu \in K$*.*

(1) *There exist* $\kappa \in \mathbb{N}$ *and a summable sequence of positive numbers* $(c_k)_k$ *such that*
- $\|T_\lambda^{n+k} S_\mu^n(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\mu \leq \lambda$*;*
- $\|T_\lambda^{n} S_\mu^{n+k}(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\lambda \leq \mu$*.*

(2) *Given* $\eta > 0$*, one can find* $\tau > 0$ *such that, for all* $n \geq 1$*,*

$$0 \leq \mu - \lambda \leq \frac{\tau}{n} \implies \|T_\lambda^n S_\mu^n(v) - v\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ *is a* G_δ*-dense subset of* X*.*

⋙ We call this the *CS-Criterion.* ⋘

Almost there

Theorem (Bayart, CJr, Menet (2022))

Assume that there exist $\alpha \in (0, 1/d)$, $\beta > \alpha d$ and $D > 0$ such that, for each $u \in \mathcal{D}$ and each compact cube K inside $\Lambda \subset \mathbb{R}^d$, the following properties hold true, where $\lambda, \mu \in K$.

(1) *There exist $C > 0$, $\kappa \in \mathbb{N}$ such that, whenever $\|\lambda - \mu\|_\infty \leq D \frac{k^\alpha}{(n+k)^\alpha}$, $n, k \in \mathbb{N}, k \geq \kappa$,*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq \frac{C}{k^\beta} \qquad \text{and} \qquad \left\|T_\lambda^n S_\mu^{n+k} u\right\| \leq \frac{C}{k^\beta}.$$

(2) *Given $\eta > 0$, there is $\tau > 0$ such that, for all $n \geq 1$,*

$$\|\lambda - \mu\| \leq \frac{\tau}{n^\alpha} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

Ideal d-dimensional generalization

Theorem (d-dimension CS-Criterion)

Assume that there is $D > 0$ such that, for each $u \in \mathcal{D}$ and each compact cube K inside $\Lambda \subset \mathbb{R}^d$, the following properties hold true, where $\lambda, \mu \in K$.

(1) *There exist $\kappa \in \mathbb{N}$ and a summable sequence of positive numbers $(c_k)_k$ such that, whenever $\|\lambda - \mu\|_\infty \leq D\frac{k^{1/d}}{(n+k)^{1/d}}$, $n, k \in \mathbb{N}, k \geq \kappa$,*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq c_k \qquad \textit{and} \qquad \left\|T_\lambda^n S_\mu^{n+k} u\right\| \leq c_k.$$

(2) *Given $\eta > 0$, there is $\tau > 0$ such that, for all $n \geq 1$,*

$$\|\lambda - \mu\| \leq \frac{\tau}{n^{1/d}} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

Ideal bi-dimensional CS-Criterion

Theorem (bi-dimensional CS-Criterion)

Assume that there is $D > 0$ such that, for each $u \in \mathcal{D}$ and each compact square K, the following properties hold true, where $\lambda, \mu \in K$.

(1) *There exist $\kappa \in \mathbb{N}$ and a summable sequence of positive numbers $(c_k)_k$ such that, whenever $\|\lambda - \mu\|_\infty \le D\sqrt{\frac{k}{n+k}}$, $n, k \in \mathbb{N}, k \ge \kappa$,*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \le c_k \qquad \textit{and} \qquad \left\|T_\lambda^n S_\mu^{n+k} u\right\| \le c_k.$$

(2) *Given $\eta > 0$, there is $\tau > 0$ such that, for all $n \ge 1$,*

$$\|\lambda - \mu\| \le \frac{\tau}{\sqrt{n}} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

A covering game

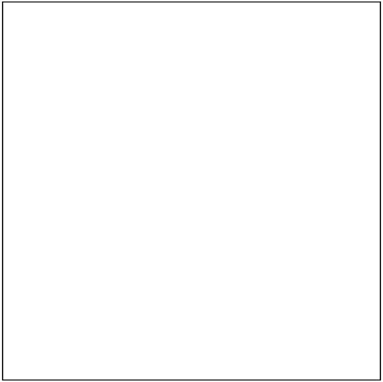
For proving the generalization of the CS-Criterion, we will choose the sequence of powers $(n_i)_i$ of the form $n_i = iN$.

One can think of this as a covering game: we are given initial values $\tau > 0$ and $N \in \mathbb{N}$ and we have to conveniently cover our parameter set by squares of sides $\frac{\tau}{\sqrt{N}}, \frac{\tau}{\sqrt{2N}}, \frac{\tau}{\sqrt{3N}}, \ldots$ and so on.

How to organize these squares can be very tricky.

A new approach

Let us explain the method for a simple case: $\frac{\tau}{\sqrt{N}} = \frac{1}{2}$. We aim to cover a unit square by square pieces of side $\frac{\tau}{\sqrt{N}}, \frac{\tau}{\sqrt{2N}}, \frac{\tau}{\sqrt{3N}}, \ldots$



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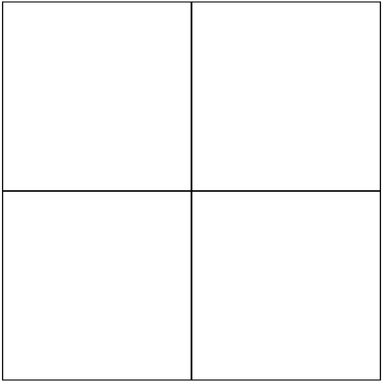

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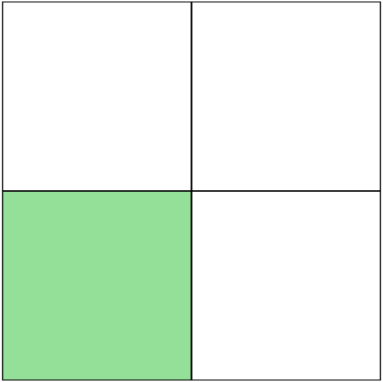

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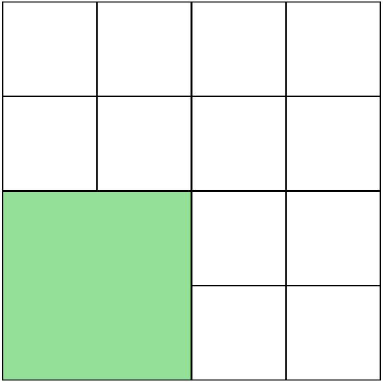

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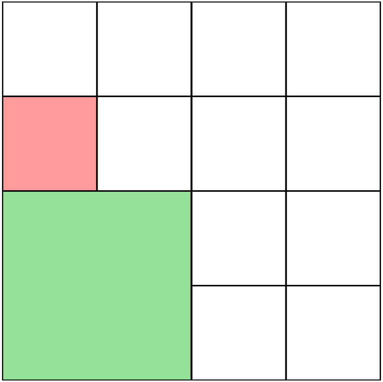

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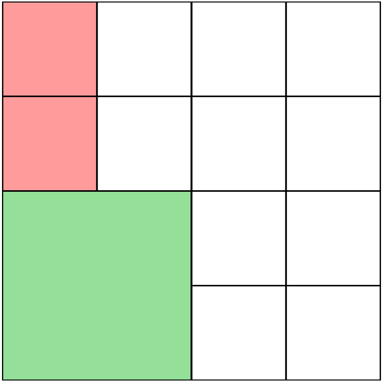

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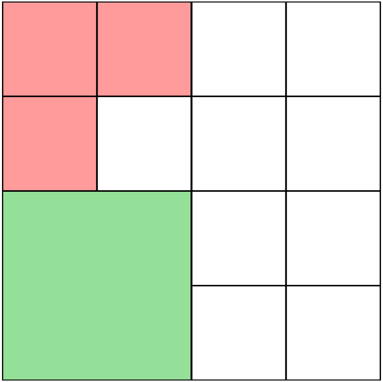

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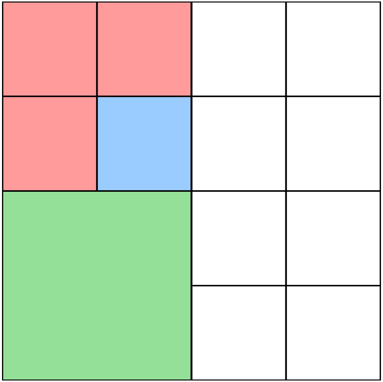

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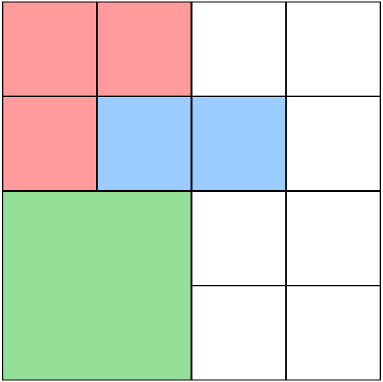

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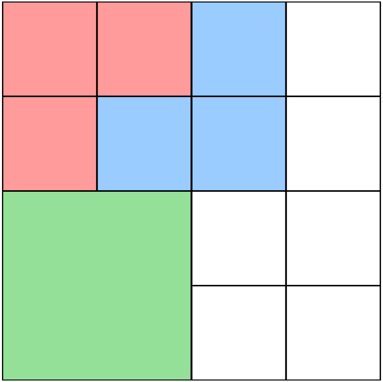

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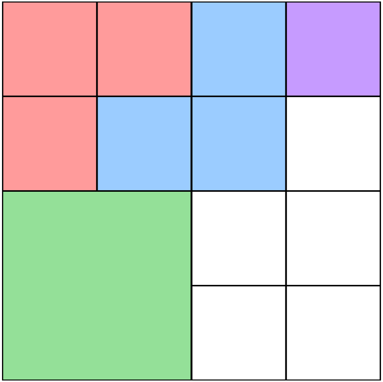

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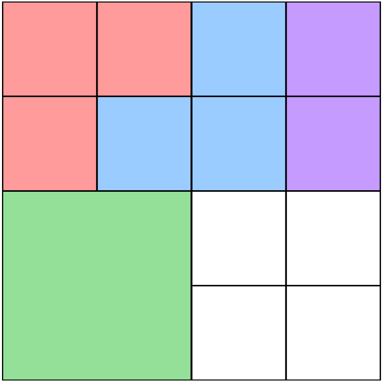

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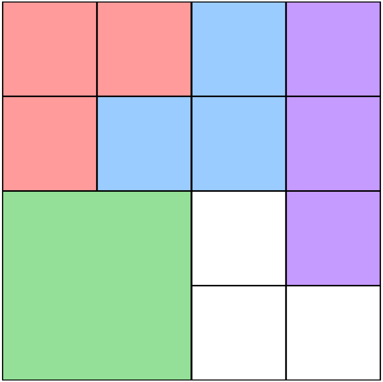

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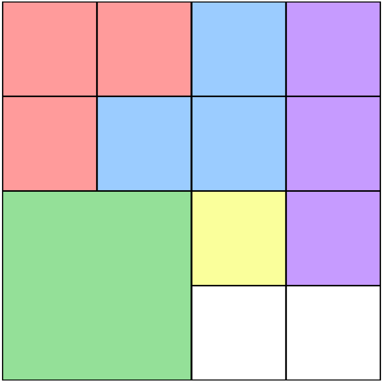

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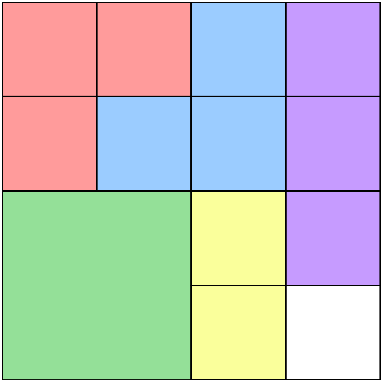

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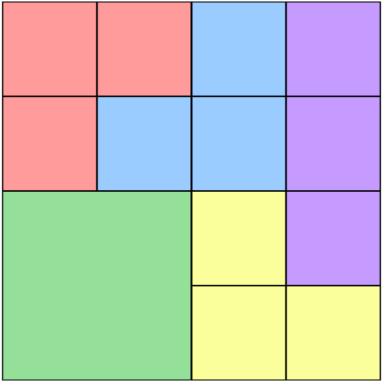
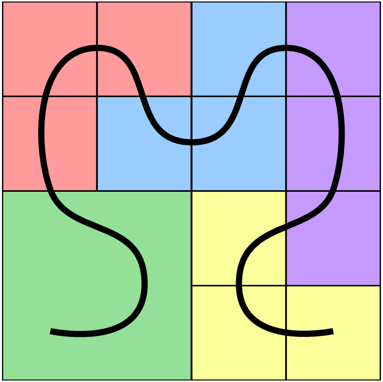
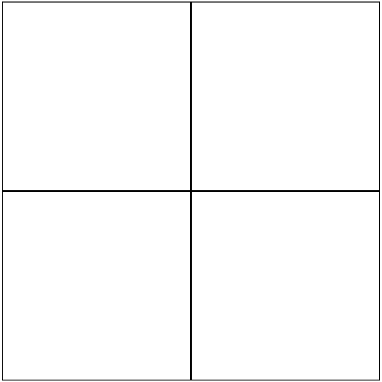


A new approach: using pseudo-Hilbert curves

Let us explain the method for a simple case: $\frac{\tau}{\sqrt{N}} = \frac{1}{2}$. We aim to cover a unit square by square pieces of side $\frac{\tau}{\sqrt{N}}, \frac{\tau}{\sqrt{2N}}, \frac{\tau}{\sqrt{3N}}, \ldots$



A covering game: round 1



Value used:

$$\frac{\tau}{\sqrt{N}} = \frac{1}{2}$$

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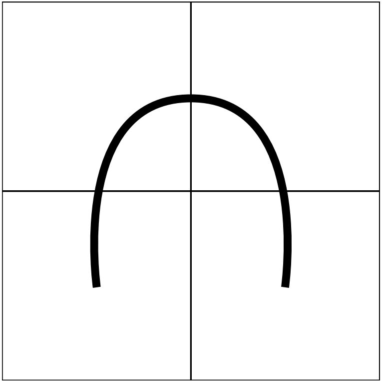

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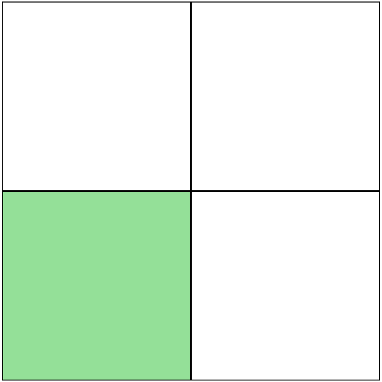

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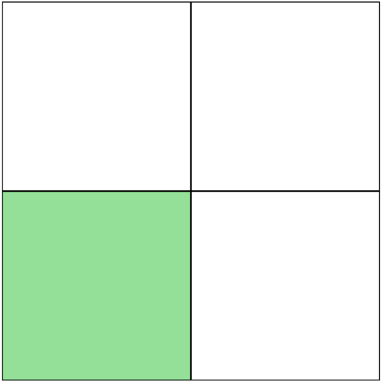

Value used:

$$\frac{\tau}{\sqrt{N}} = \frac{1}{2}$$

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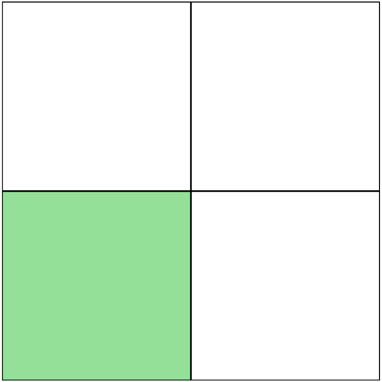
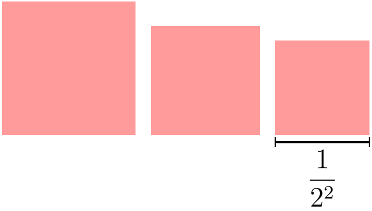
Round 1: victory!

A covering game: round 2

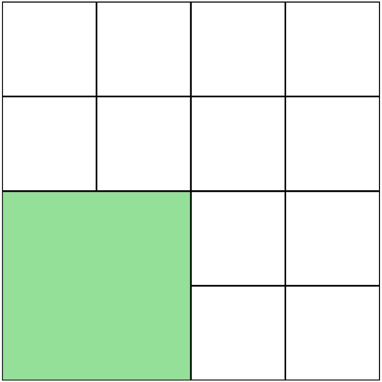


Values used:

$$\frac{\tau}{\sqrt{2N}} > \frac{\tau}{\sqrt{3N}} > \frac{\tau}{\sqrt{4N}} = \frac{1}{2^2}$$



A covering game: round 2



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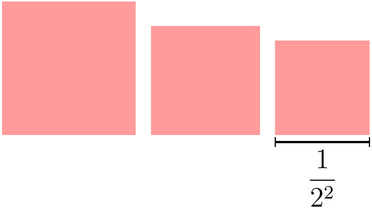

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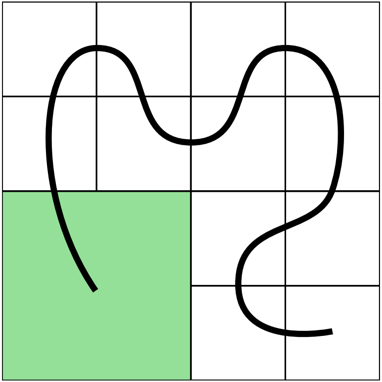

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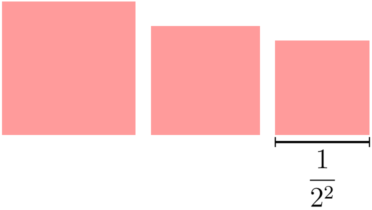

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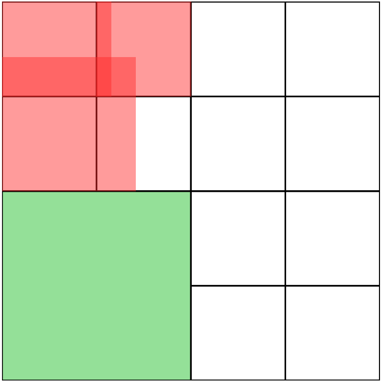

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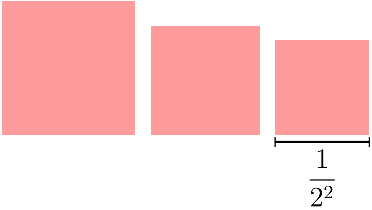

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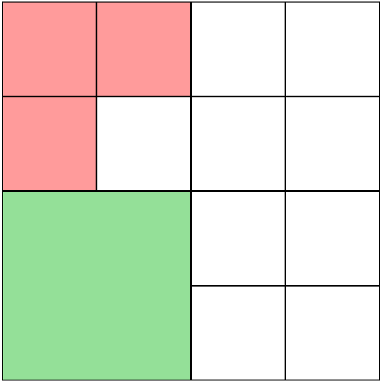

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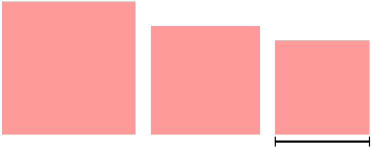
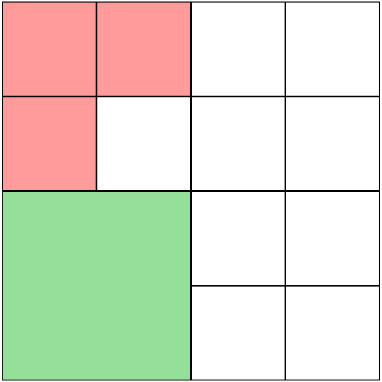


$$\frac{1}{2^2}$$

A covering game: round 2



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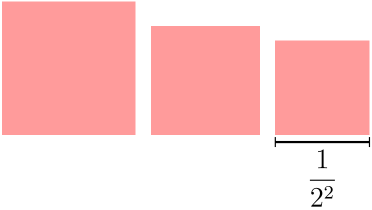
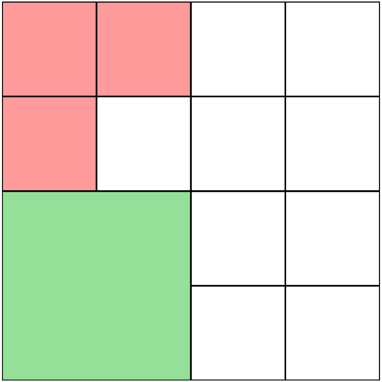


Round 2: victory!

A covering game: round 3



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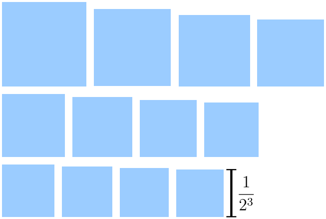
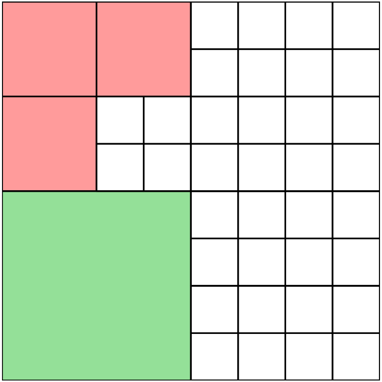


A covering game: round 3



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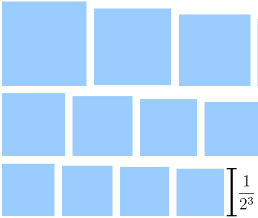

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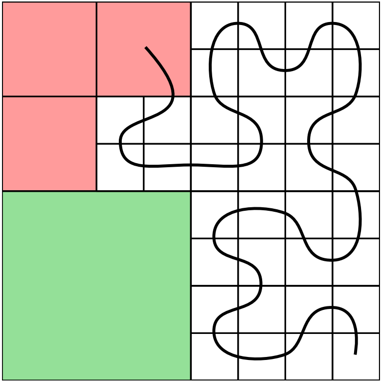

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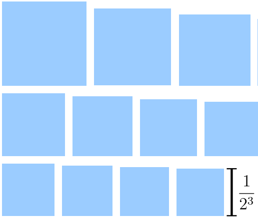
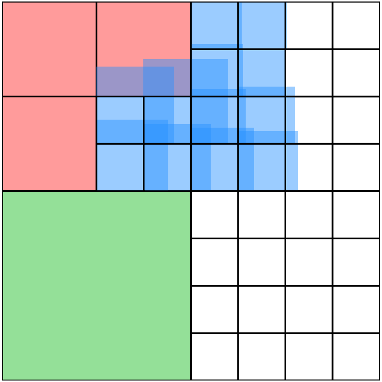
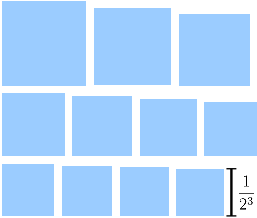


A covering game: round 3

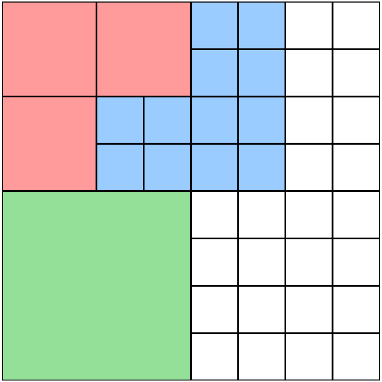


Values used:

$$\frac{\tau}{\sqrt{5N}} > \frac{\tau}{\sqrt{6N}} > \cdots > \frac{\tau}{\sqrt{16N}} = \frac{1}{2^3}$$



A covering game: round 3



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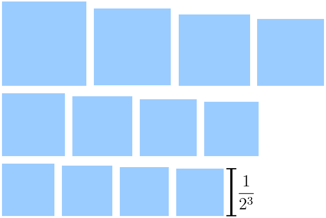
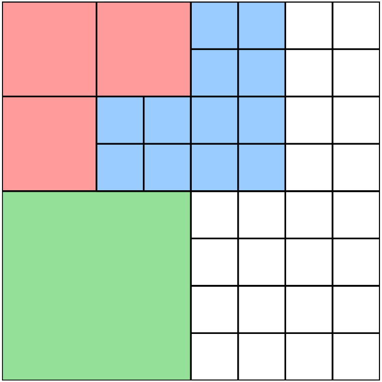


A covering game: round 3



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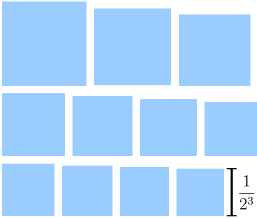
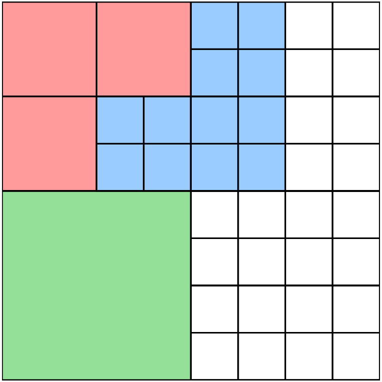


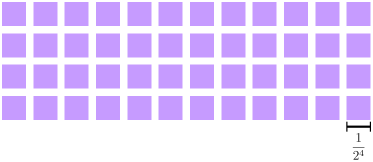
Round 3: victory!

A covering game: round 4

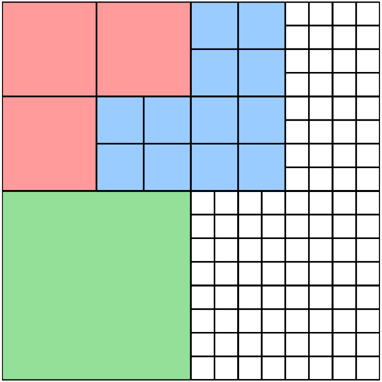


Values used:

$$\underbrace{\frac{\tau}{\sqrt{17N}} > \cdots > \frac{\tau}{\sqrt{64N}}}_{\text{to make } 48=3\times 4^2 \text{ pieces}} = \frac{1}{2^4}$$



A covering game: round 4



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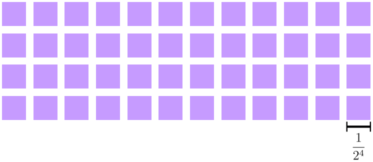
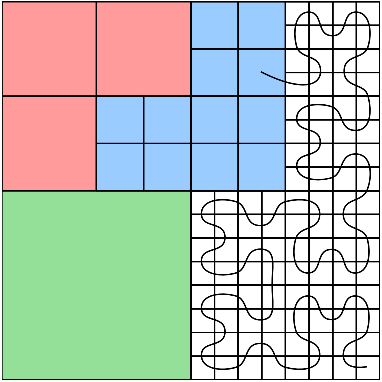


A covering game: round 4



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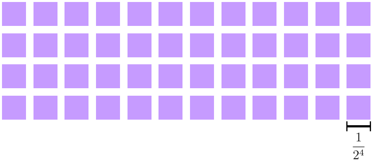

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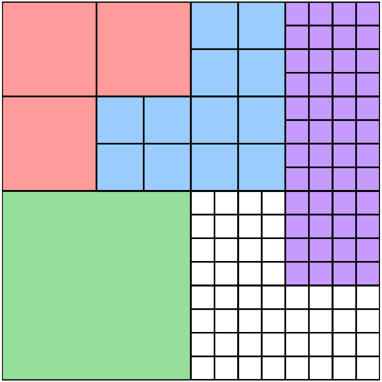

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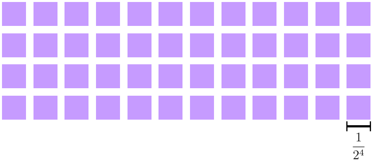

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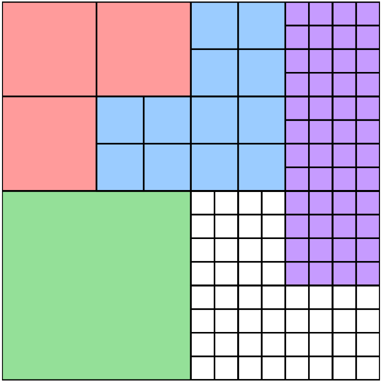

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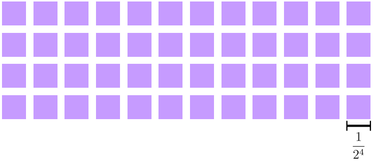
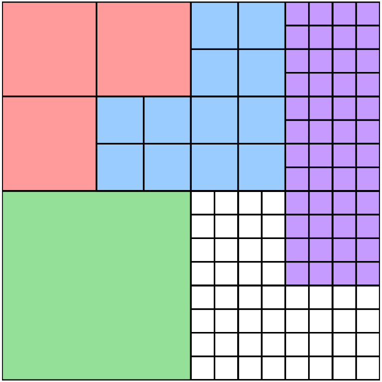


Round 4: victory!

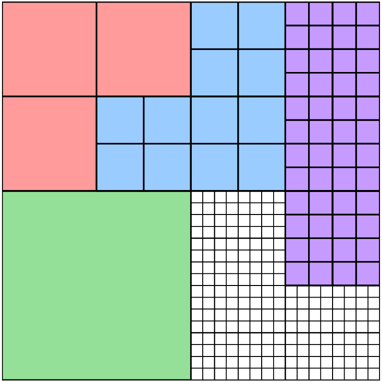
A covering game: round 5



Values used:

$$\underbrace{\frac{\tau}{\sqrt{65N}} > \cdots > \frac{\tau}{\sqrt{256N}}}_{\text{to make } 192=3\times 4^3 \text{ pieces}} = \frac{1}{2^5}$$

A covering game: round 5



Values used:

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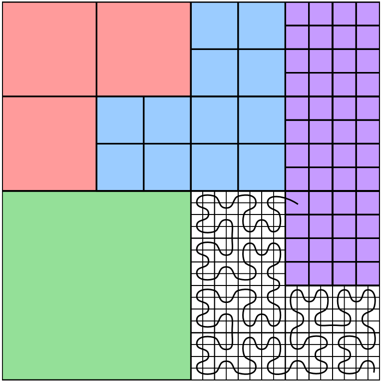

Values used:

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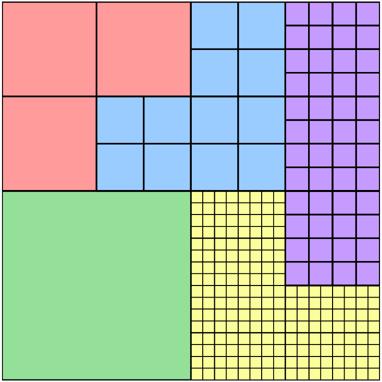

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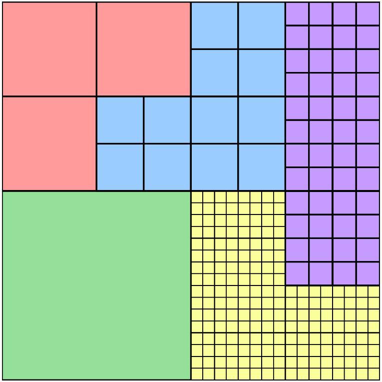

Values used:

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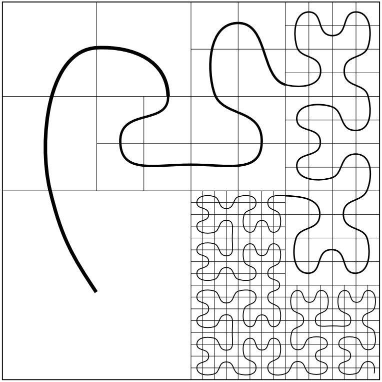
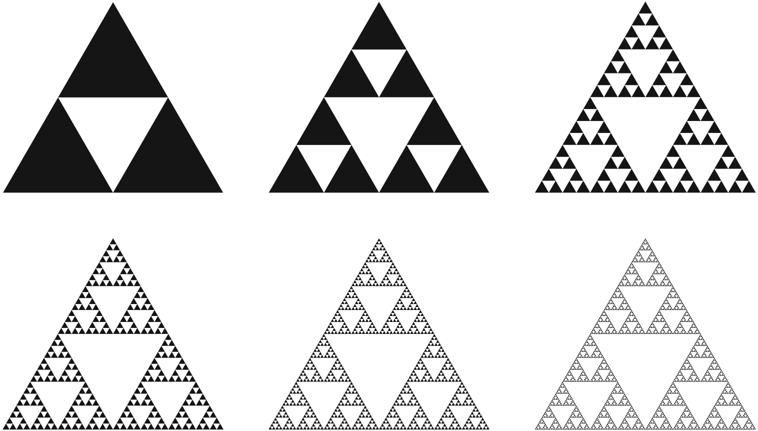
Round 5: victory. The game is over.

How the enumeration of our covering looks like



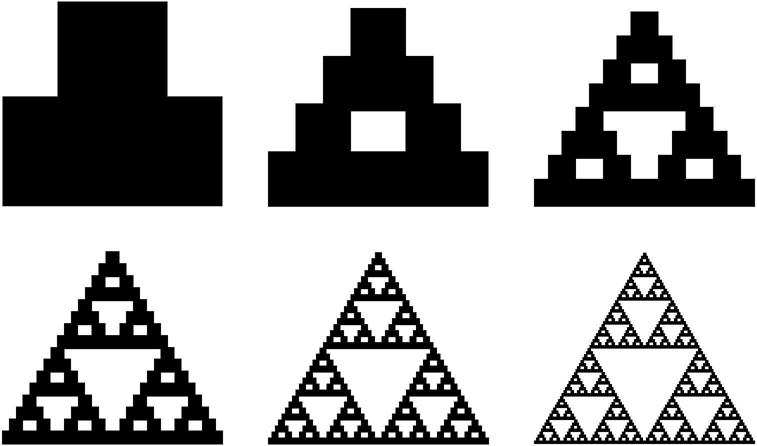
Ordering of $\Lambda_1, \Lambda_2, \ldots, \Lambda_{256}$ such that $K \subset \bigcup_{i=1}^{256} \Lambda_i$

Sierpiński gasket



Obtaining the Sierpiński gasket through triangles

Sierpiński gasket



Obtaining the Sierpiński gasket through squares

Dimension of the gasket

Let Γ be a unitary Sierpiński gasket inside $[0,+\infty)^2$. The similarity dimension (= Hausdorff dimension) of the gasket is $\dim_{\mathcal{H}} = \frac{\log 3}{\log 2}$.

Up to now, we knew how to state a generalization of the CS-Criterion for all $\alpha < \frac{1}{\dim_{\mathcal{H}} \Gamma} = \frac{\log 2}{\log 3}$.

Applying the new approach to the gasket

The limit case we aim to address is $\alpha = \frac{1}{\frac{\log 3}{\log 2}} = \frac{\log 2}{\log 3}$.

Once again, we are given values $\tau > 0$, $N \in \mathbb{N}$ and we want to construct a covering by squares of sides

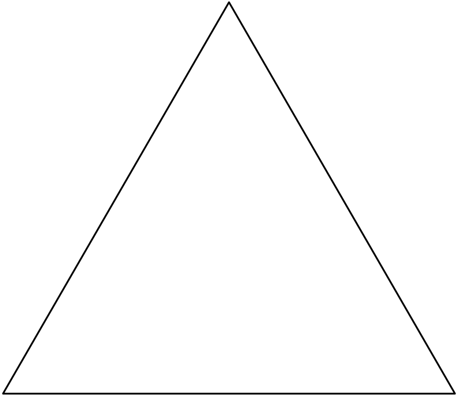
$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}}, \frac{\tau}{(2N)^{\frac{\log 2}{\log 3}}}, \frac{\tau}{(3N)^{\frac{\log 2}{\log 3}}}, \ldots$$

to make pieces for our covering.

The covering game of the gasket: grouping method

Again, let us consider the simple case

$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}} = \frac{1}{2}.$$



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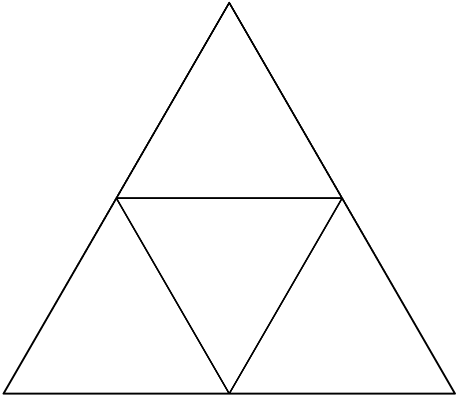

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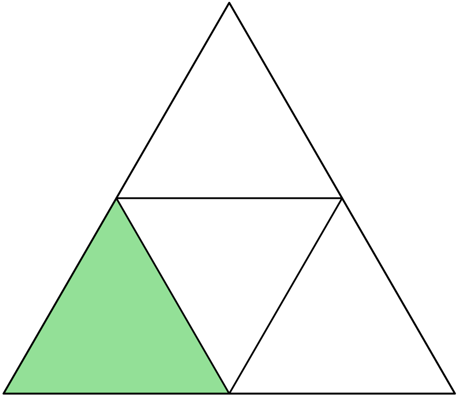

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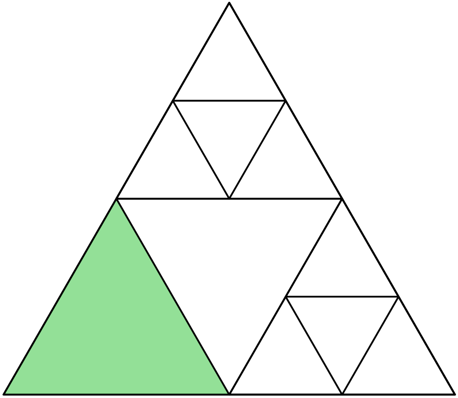

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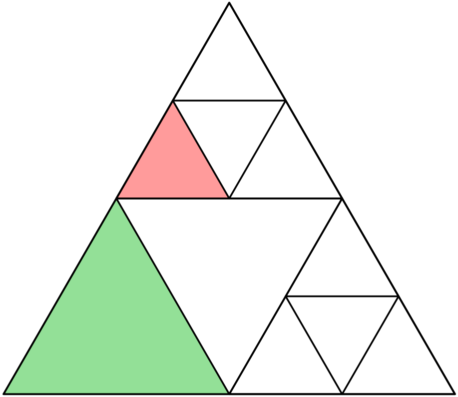

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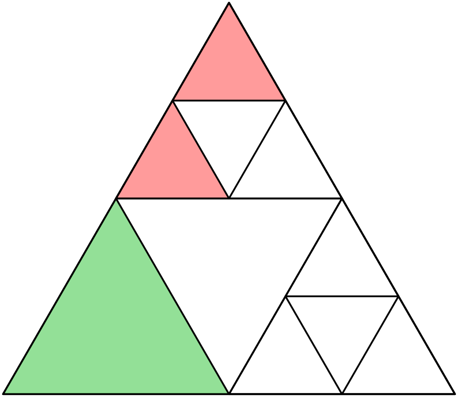

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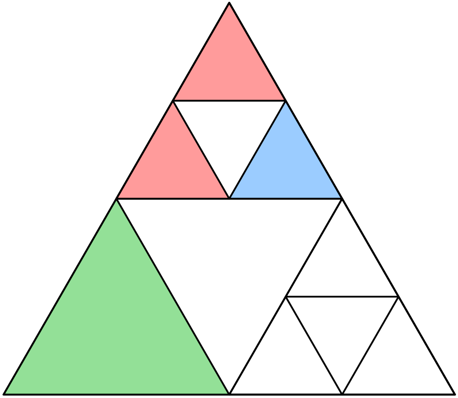

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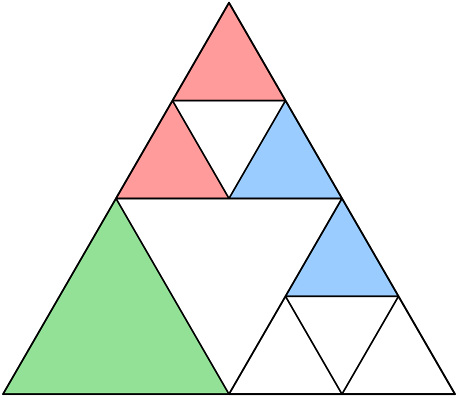

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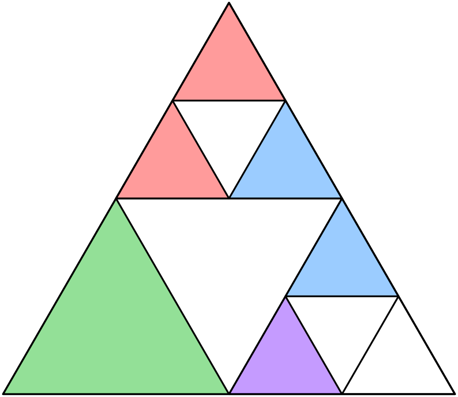

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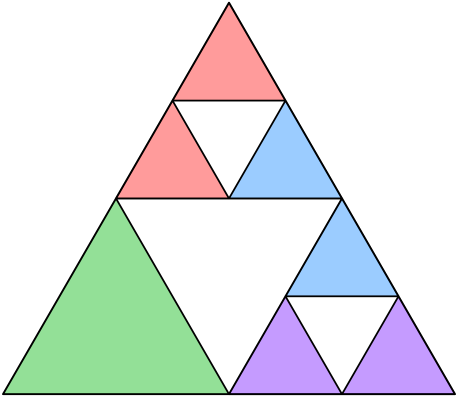

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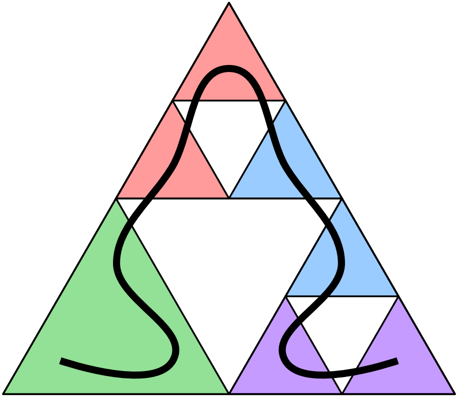
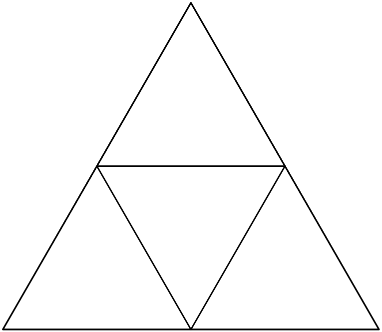


The covering game of the gasket: round 1



Value used:

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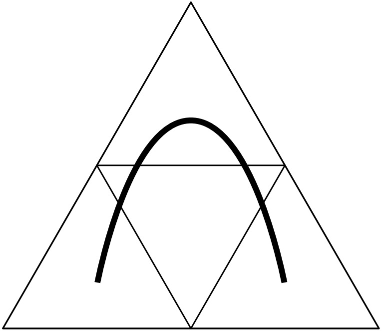
The covering game of the gasket: round 1



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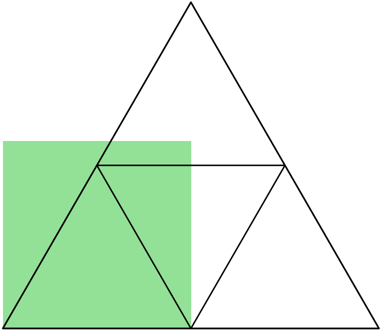
A covering game: round 1



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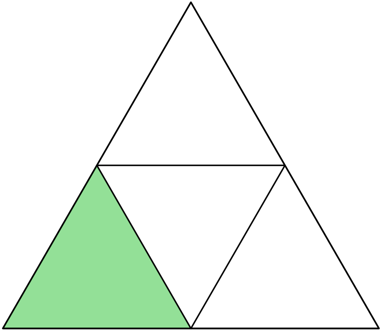

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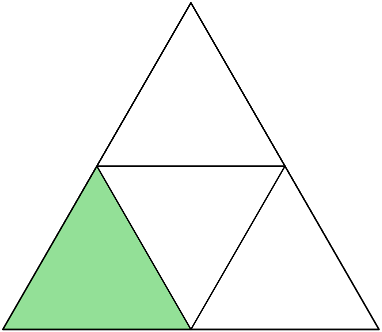

Value used:

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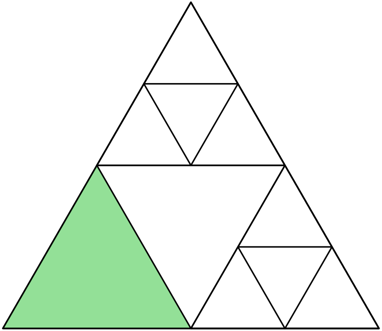
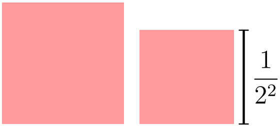
Round 1: victory!

The covering game of the gasket: round 2

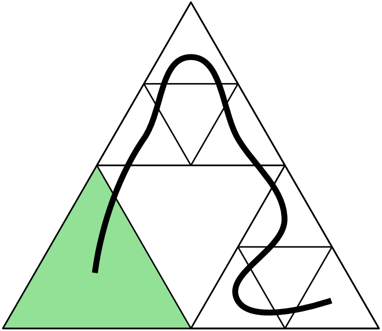


Values used:

$$\frac{\tau}{(2N)^{\frac{\log 2}{\log 3}}} > \frac{\tau}{(3N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^2}$$



The covering game of the gasket: round 2



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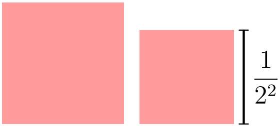
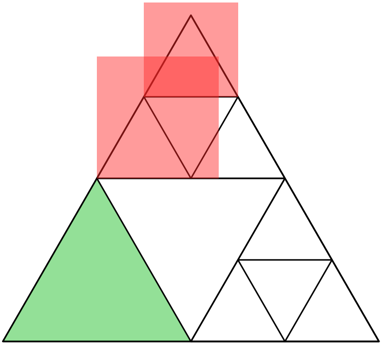


A covering game: round 2



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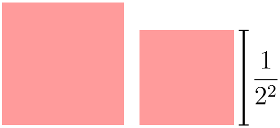

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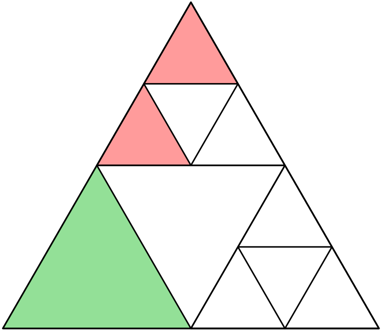

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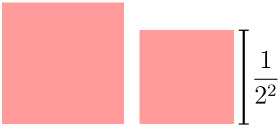

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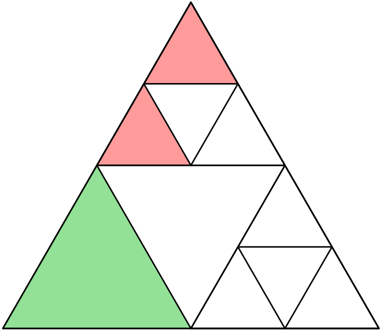

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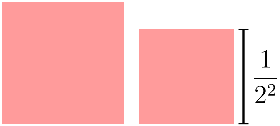
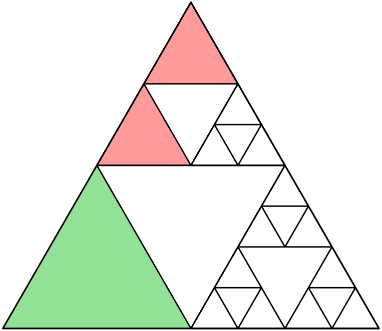


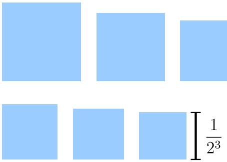
Round 2: victory!

The covering game of the gasket: round 3

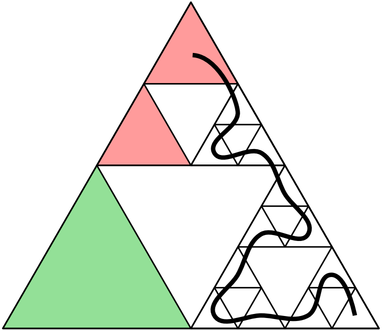


Values used:

$$\frac{\tau}{(4N)^{\frac{\log 2}{\log 3}}} > \cdots > \frac{\tau}{(9N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^3}$$

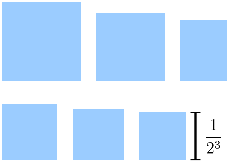


The covering game of the gasket: round 3

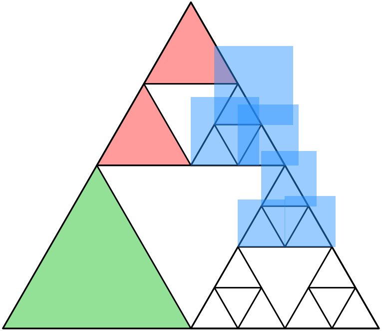


Values used:

$$\frac{\tau}{(4N)^{\frac{\log 2}{\log 3}}} > \cdots > \frac{\tau}{(9N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^3}$$



A covering game: round 3



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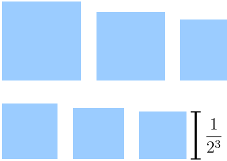
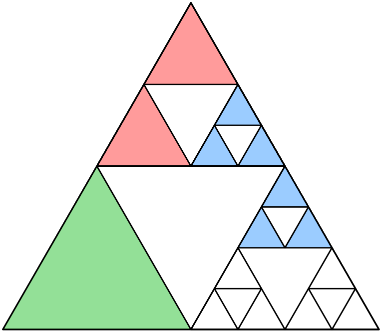


A covering game: round 3



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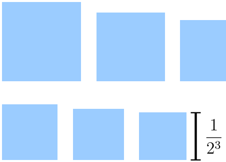
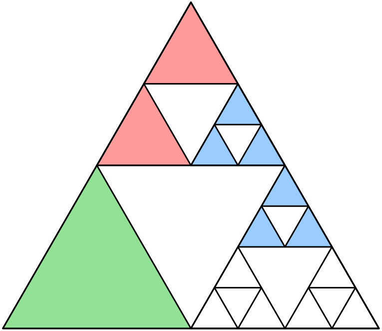


A covering game: round 3



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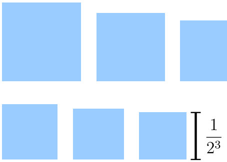
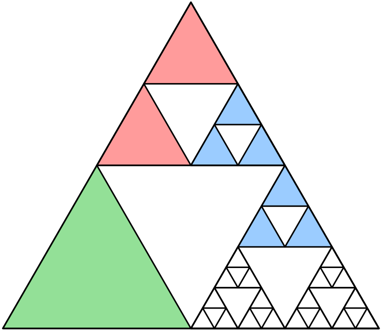


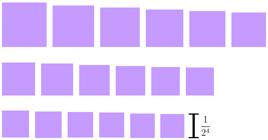
Round 3: victory!

The covering game of the gasket: round 4

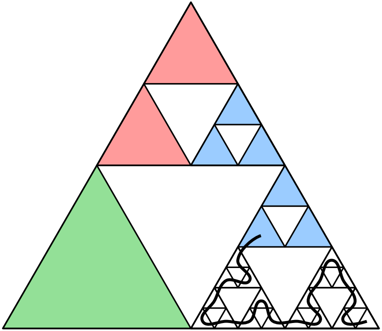


Values used:

$$\frac{\tau}{(10N)^{\frac{\log 2}{\log 3}}} > \cdots > \frac{\tau}{(27N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^3}$$



The covering game of the gasket: round 4



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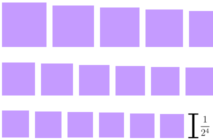
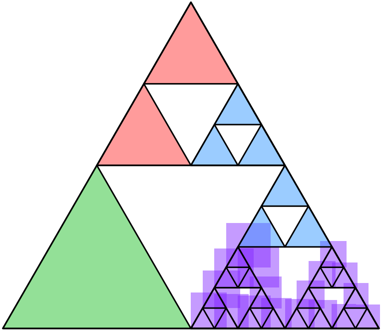


A covering game: round 4



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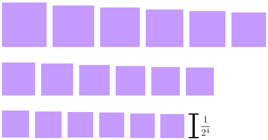
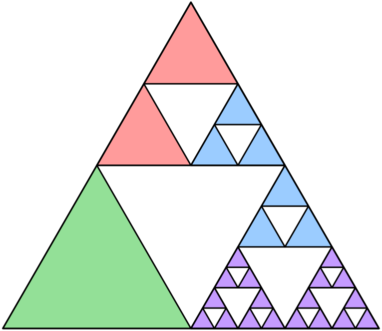


$\frac{1}{2^4}$

A covering game: round 4



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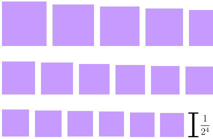
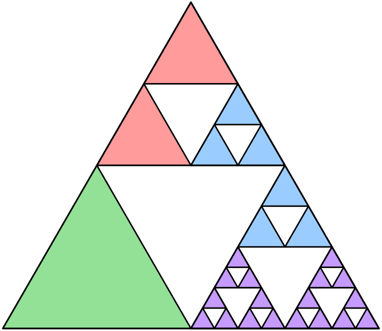


A covering game: round 4



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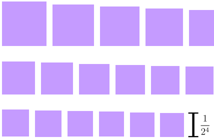
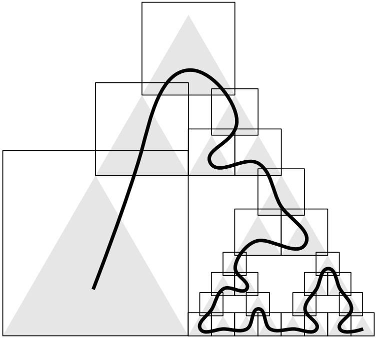


Round 4: victory! The game is over.

How the enumeration of our covering looks like



Ordering of $\Lambda_1, \Lambda_2, \ldots, \Lambda_{27}$ such that $\Gamma \subset \bigcup_{i=1}^{27} \Lambda_i$

Koch snowflake

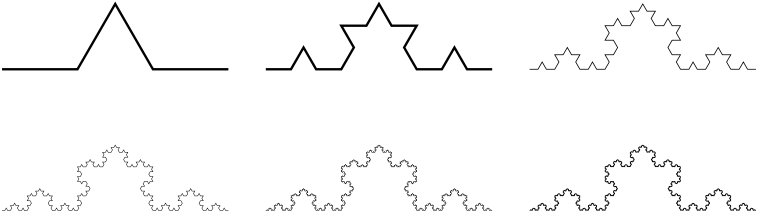


Figure : The von Koch curve forms the well known Koch snowflake. It has 4 similarities and contraction ration $\frac{1}{3}$. Its has Hausdorff dimension $\frac{\log(4)}{\log(3)}$.

We can apply our method by making groups of 3 patches of parts.

Minkowski sausage

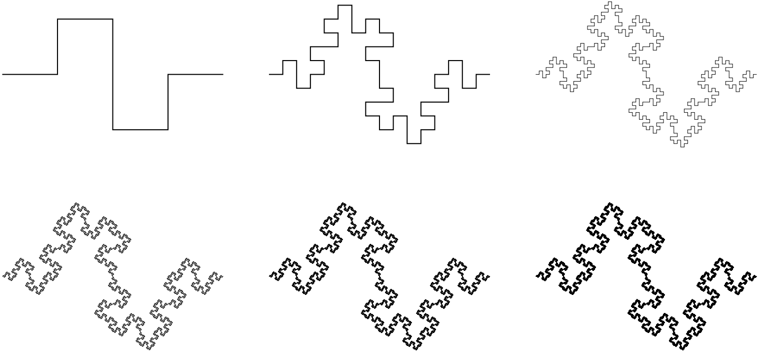


Figure : The Minkowski sasusage has 8 similarities with $\frac{1}{4}$ contraction ration. Its Hausdorff dimension is 1.5.

We can apply our method by making groups of 7 patches of parts.

A new concept of dimension?

Definition. Let $d \geq 1$ and $\Lambda \subset \mathbb{R}^d$ be compact. We say that Λ has *homogeneously ordered box dimension at most* $\gamma \in (0, d]$ if there exist $r \in [\![2, +\infty[\![$, $\rho > 0$ and, for all $m \geq 1$, a compact covering $(\Lambda_{\mathbf{i}})_{\mathbf{i} \in I_r^m}$ of Λ satisfying the following.

(i) For all $\mathbf{i} \in I_r^m$,

$$\operatorname{diam}(\Lambda_{\mathbf{i}}) \leq \rho \left(\frac{1}{r^{1/\gamma}} \right)^m.$$

(ii) For all $\mathbf{i} = (i_1, ..., i_m, i_{m+1}) \in I_r^{m+1}$,

$$\Lambda_{i_1, \ldots, i_{m+1}} \subset \Lambda_{i_1, \ldots, i_m}.$$

(iii) For all $\mathbf{i} = (i_1, ..., i_m) \in I_r^m$ and all $j \in \{2, ..., r\}$,

$$\Lambda_{\mathbf{i}, j-1, r} \cap \Lambda_{\mathbf{i}, j, 1} \neq \varnothing.$$

The *homogeneously ordered box dimension of* Λ (HBD° for short) is the smallest number γ such that Λ has homogeneously ordered box dimension at most γ. We will denote this number by $\dim_{HB}^{\circ}(\Lambda)$.

A full generalization of the CS-criterion

Theorem (Generalized Costakis-Sambarino Criterion)

Let $d \in \mathbb{N}$ and $\gamma > 0$ and let Λ be a compact subset of $\mathbb{R}^d$ having homogeneously ordered box dimension at most γ. Assume that there is $D > 0$ such that, for all $u \in \mathcal{D}$ and all compact $K \subset \Lambda$, the following properties hold true, where all parameters λ, μ belong to K.

(1) *There exist $\kappa \in \mathbb{N}$ and a sequence of positive numbers $(c_k)_{k\in\mathbb{N}}$ such that $\sum_{k=\kappa}^{\infty} c_k < \infty$ and, whenever $\|\lambda - \mu\|_\infty \leq D\frac{k^{1/\gamma}}{(n+k)^{1/\gamma}}$, with $n, k \in \mathbb{N}_0$ and $k \geq \kappa$, we have*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq c_k \qquad \text{and} \qquad \left\|T_\lambda^n S_\mu^{n+k} u\right\| \leq c_k.$$

(2) *Given $\eta > 0$, one can find $\tau > 0$ such that, for all $n \in \mathbb{N}$,*

$$\|\lambda - \mu\|_\infty \leq \frac{\tau}{n^{1/\gamma}} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

Applications

Let $d \geq 1$ and $\Gamma \subset \mathbb{R}^d$ compact. We write $\lambda = (x_1, \ldots, x_d)$ for any $\lambda \in \mathbb{R}^d$. Consider $\alpha \in (0, 1/d]$ and a family $\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big)_{\lambda \in \Gamma}$ induced by weights $\big(w(x)\big)_{x \in \mathbb{R}}$ satisfying

$$w_1(x) \cdots w_d(x) \approx \exp(xn^\alpha).$$

Theorem

If Γ has HBO° at most $\gamma \leq d$ and $0 < \alpha \leq 1/\gamma$, then

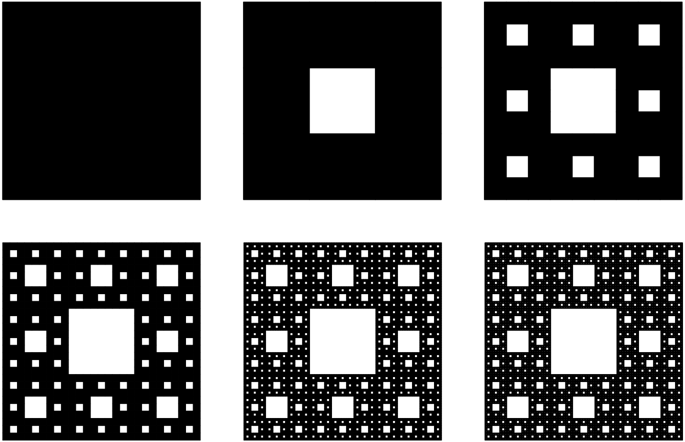
$$\bigcap_{\lambda \in \Gamma} HC\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big) \neq \varnothing.$$

Corollary

If Γ is a β-Hölder curve and $0 < \alpha \leq \beta$, then

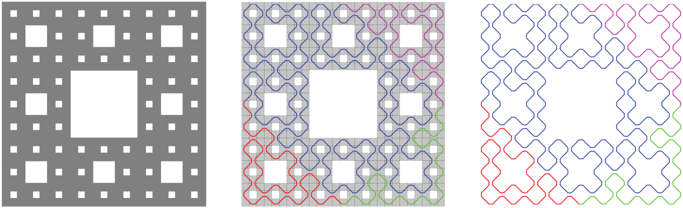
$$\bigcap_{\lambda \in \Gamma} HC\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big) \neq \varnothing.$$

IFS, GIFS and optimal parametrizations



Construction of the Sierpiński carpet.

IFS, GIFS and optimal parametrizations



Ordering the carpet (Rao and Zhang, 2016).

Some open questions

• $\sum \frac{1}{F(n)^2} = +\infty \implies \bigcap_{x,y>0} HC(B_{w(x)} \times B_{w(y)}) \neq \varnothing$?

• For $\mathcal{C} \subset \mathbb{R}_+^2$ the Cantor dust with dissection ratio $1/4$,

$$\bigcap_{(x,y)\in\mathcal{C}} e^x B \times e^y B \neq \varnothing?$$

• Can we, more generally, generalize our result for totally disconnected fractals?

• Can we adapt optimally our technique to self-similar fractals with non-uniform contraction ratio?

• For $\mathcal{T} \subset \mathbb{R}_+^2$ the 1-dimensional Takagi curve,

$$\bigcap_{(x,y)\in\mathcal{T}} e^x B \times e^y B \neq \varnothing?$$