Corrigé CC1 algèbre 2

Exercice 1

$$(S)\begin{cases} x + y + (m+1)z = 2 \\ mx + 2my + mz = m+1 \\ x + (m+1)y - 5z = 5 \end{cases} \qquad \begin{matrix} \\ L_2 \leftarrow L_2 - mL_1 \\ L_3 \leftarrow L_3 - L_1 \end{matrix}$$

$$\Longleftrightarrow \begin{cases} x + y + (m+1)z = 2 \\ my - m^2 z = -m+1 \\ my + (-m-6)z = 3 \end{cases} \qquad \begin{matrix} \\ \\ L_3 \leftarrow L_3 - L_2 \end{matrix}$$

$$\Longleftrightarrow \begin{cases} x + y + (m+1)z = 2 \\ my - m^2 z = -m+1 \\ \underbrace{(m^2-m-6)}_{=(m-3)(m+2)} z = 2+m \end{cases}$$

$\ast$ Si $m = 3$

$$(S) \Longleftrightarrow \begin{cases} x + y + 4z = 2 \\ 3y - 9z = -2 \\ 0 = 5 \end{cases}$$

Le système est incompatible

* Si $m = -2$

$$(S) \iff \begin{cases} x + y - 3z = 2 \\ -2y - 4z = 3 \\ 0 = 0 \end{cases}$$

$$\iff \begin{cases} y = \frac{1}{2}(-4z - 3) = -2z - \frac{3}{2} \\ x = 2 - y + 3z = 2 + 2z + \frac{3}{2} + 3z = \frac{7}{2} + 3z \end{cases}$$

Dimension 1, rang 2.

* Si $m = 0$

$$(S) \iff \begin{cases} x + y + z = 2 \\ 0 = 1 \\ -6z = 2 \end{cases}$$

Le système est incompatible.

* Si $m \notin \{3, -2, 0\}$

$$(S) \iff \begin{cases} z = \frac{1}{m-3} \\ y = \frac{1}{m}\left(-m + 1 + m^2 z\right) = -1 + \frac{1}{m} + \frac{m}{m-3} = \frac{4m-3}{m(m-3)} \\ x = 2 - y - (m+1)z = 2 - \frac{4m-3}{m(m-3)} - \frac{m+1}{m-3} = \frac{m^2 - 11m + 3}{m(m-3)} \end{cases}$$

Dimension 0, rang 3.

Exercice 2

$$\mathrm{rg}\begin{pmatrix} 3 & 1 & 2 & -1 \\ -4 & 3 & 0 & 1 \\ 2 & 5 & 4 & -1 \end{pmatrix} \qquad C_1 \longleftrightarrow C_4$$

$$= \mathrm{rg}\begin{pmatrix} -1 & 1 & 2 & 3 \\ 1 & 3 & 0 & -4 \\ -1 & 5 & 4 & 2 \end{pmatrix} \qquad \begin{array}{l} L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$= \mathrm{rg}\begin{pmatrix} -1 & 1 & 2 & 3 \\ 0 & 4 & 2 & -1 \\ 0 & 4 & 2 & -1 \end{pmatrix} \qquad L_3 \leftarrow L_3 - L_2$$

$$= \mathrm{rg}\begin{pmatrix} -1 & 1 & 2 & 3 \\ 0 & 4 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} C_2 \leftarrow C_2 + 4\,C_4 \\ C_3 \leftarrow C_3 + 2\,C_4 \end{array}$$

$$= \mathrm{rg}\begin{pmatrix} -1 & 13 & 8 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} C_2 \leftarrow C_2 + 13\,C_1 \\ C_3 \leftarrow C_3 + 8\,C_1 \\ C_4 \leftarrow C_4 + 3\,C_1 \end{array}$$

$$= \mathrm{rg}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 2$$

Exercice 3

1- $M^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$

$M^3 = M^2 M = -M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$M^4 = M^2 M^2 = I_2$.

2- On a $M(-M) = I_2$ donc M est inversible et $M^{-1} = -M$

Exercice 4

Soient $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$.

$$B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\iff \begin{cases} x + y = a \\ x + y + z = b \\ y + z = c \end{cases} \qquad L_2 \leftarrow L_2 - L_1$$

$$\iff \begin{cases} x + y = a \\ z = b - a \\ y + z = c \end{cases}$$

$$\iff \begin{cases} z = b - a = -a + b \\ y = c - z = c - b + a = a - b + c \\ x = a - y = b - c \end{cases}$$

$$\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}_{B^{-1}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$