Common Hypercyclic Algebras

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Introduction

In the following, X denotes a Fréchet space and T a continuous linear operator on X. We say that a $x \in X$ is a **hypercyclic vector** for T when its orbit $Orb(x, T) := \{T^n x : n \geq 0\}$ is dense in X. If such a vector exists we say that T is a **hypercyclic operator** on X. The set of hypercyclic vectors for an operator T is denoted by $HC(T)$.

Introduction

Some classical examples of hypercyclic operators are the derivative operator $D : f \mapsto f'$ acting on $H(\mathbb{C})$ (MacLane operator), the multiples of the backward $\lambda B, |\lambda| > 1$, acting on ℓ_p, $1 \leq p < \infty$, or c_0 (Rolewicz operators) and the translation operators $T_a f : z \mapsto f(z + a), a \neq 0$, acting on $H(\mathbb{C})$ (Birkoff operators).

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Linear subspaces of hypercyclic vectors

It is known that, whenever T is hypercyclic, the set $HC(T)$ is a dense G_δ-set. Furthermore, $HC(T) \cup \{0\}$ always contains a non-trivial linear subspace. More interesting questions and criteria appear when we look for a closed and infinite dimensional subspace, the so called **hypercyclic subspaces**. For example the unilateral shift $2B$ does not admit a hypercyclic subspace, on the other hand any hypercyclic bilateral backward shift on $\ell_2(\mathbb{Z})$ does.

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Fréchet algebras

When X is an algebra it is natural to ask whether or not $HC(T) \cup \{0\}$ contains a non-trivial subalgebra. Such an structure will be called a **hypercyclic algebra** for T. Regarding sequence spaces, two products are commonly considered : the convolution (or Cauchy) product and the coordinatewise product. For the first one, $H(\mathbb{C})$ and ℓ_1 are Fréchet sequence algebras. For the second, all the spaces c_0, $H(\mathbb{C})$ and ℓ_p with $1 \leq p < \infty$ are Fréchet sequence algebras.

The first negative result for the existence of hypercyclic algebras was obtained by Aron et. al. [1]. The first positive result was obtained independently by Shkarin [2] and by Bayart and Matheron [3] for D on $H(\mathbb{C})$.

Fréchet algebras

The approach of Bayart & Matheron [3] relies on a Baire argument and was based on the following lemma.

Lemma 1. Bayart & Matheron (2009)

Let X be a Fréchet algebra and T be a continuous operator on X. Suppose that, for all $m \in \mathbb{N}$ and all $U, V, W \subset X$ open and non-empty, with $0 \in W$, there exists $N \in \mathbb{N}$ and $u \in U$ satisfying

$$\begin{cases} T^N u^n \in W, \text{ for } n = 1, ..., m-1; \\ T^N u^m \in V. \end{cases}$$

Then T admits a hypercyclic algebra.

Using this result we now know that the following operators admit a hypercyclic algebra (for the Cauchy product) : λD on $H(\mathbb{C})$, for all $\lambda > 0$; and λB on ℓ_1, for all $|\lambda| > 1$.

Obs. : Actually the same holds for the coordinatewise product !

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Families of operators

Considering all these multiples of the backward shift as a family, Abakumov and Gordon (see [4]) have shown that $(\lambda B)_{\lambda>1}$ admits a **common hypercyclic vector**, that is, that $\bigcap_{\lambda>1} HC(\lambda B) \neq \emptyset$. The same is true for the family $(\lambda D)_{\lambda>0}$.

Could we have a common hypercyclic algebra ?

...The first step would be to find a new criterion...

Families of operators - New criterion

Using some ideas from the proof of the Costakis-Sambarino Criterion (see [5]) we reached the following.

Lemma

Let $T \in \mathcal{L}(X)$, where X is an F-algebra, and let $\Gamma = [a, b]$ with $0 < a < b$. Suppose that, for each pair (U, V) of nonempty open sets in X, each 0-neighborhood O in X and each $m \in \mathbb{N}$, there exist $q \in \mathbb{N}$, a partition $a = \lambda_0 < \lambda_1 < \cdots < \lambda_q = b$ of Γ, positive integers $N_1, \ldots, N_q$ and a point $u \in U$ such that, for each $i \in \{1, \ldots, q\}$ and all $\lambda \in [\lambda_{i-1}, \lambda_i]$, we have

- $(\lambda T)^{N_i}(u^j) \in O$ for $j \in \{1, \ldots, m-1\}$;
- $(\lambda T)^{N_i}(u^m) \in V$.

Then $(\lambda T)_{\lambda \in \Gamma}$ admits a common hypercyclic algebra.

Families of operators - New generalized criterion

The following is a more general version.

Lemma

Let $T \in \mathcal{L}(X)$, where X is an F-algebra, and let $\Gamma = [a, b]$ with $0 < a < b$. Suppose that, for each pair (U, V) of nonempty open sets in X, each 0-neighborhood O in X and each $\mathcal{I} \in \mathcal{P}_f(\mathbb{N}) \backslash \{\emptyset\}$, there exist $m \in \mathcal{I}$, $q \in \mathbb{N}$, a partition $a = \lambda_0 < \lambda_1 < \cdots < \lambda_q = b$ of Γ, positive integers $N_1, \ldots, N_q$ and a point $u \in U$ such that, for each $i \in \{1, \ldots, q\}$ and all $\lambda \in [\lambda_{i-1}, \lambda_i]$, we have

- $(\lambda T)^{N_i}(u^j) \in O$ for $j \in \mathcal{I} \backslash \{m\}$;
- $(\lambda T)^{N_j}(u^m) \in V$.

Then $(\lambda T)_{\lambda \in \Gamma}$ admits a common hypercyclic algebra.

Families of operators - Practical criterion for a vector

To be more precise... We actually have a practical criterion that guarantees the existence of a **common hypercyclic vector** (see [6]) for a family $(T_\lambda)_{\lambda\in\Lambda}$ of weighted shifts $T_\lambda = B_{w(\lambda)}$.

Theorem. Bayart & Matheron (2007)

Let X a Fréchet sequence space with an unconditional basis (e_n). Consider a family of bounded wheighted shifts $(T_\lambda)_{\lambda\in\Lambda}$ acting on X and indexed by a σ-compact set Λ such that the map $(x,\lambda)\mapsto T_\lambda(x)$ is continuous from $\Lambda\times X$ into X. Then $(T_\lambda)_{\lambda\in\Lambda}$ admits a common hypercyclic vector as soon as the following properties are satisfied.

- All functions $\log(w_n)$ are nondecreasing and Lipschitz on compacts sets with uniformly bounded Lipschitz constants ;
- All series $\sum_n \frac{1}{w_1(\lambda)\cdots w_n(\lambda)} e_n$ are convergent.

Could we get something like this for algebras ? ?

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Common Hypercyclic Algebras - The coordinatewise product

For the coordinatewise product an analogous statement holds true.

Theorem

Let X a Fréchet sequence space with an unconditional basis (e_n). Consider a family of bounded wheighted shifts $(T_\lambda)_{\lambda\in\Lambda}$ acting on X and indexed by a σ-compact set Λ such that the map $(x,\lambda)\mapsto T_\lambda(x)$ is continuous from $\Lambda\times X$ into X. Then $(T_\lambda)_{\lambda\in\Lambda}$ admits a common hypercyclic algebra as soon as the following properties are satisfied.

- All functions $\log(w_n)$ are nondecreasing and Lipschitz on compacts sets with uniformly bounded Lipschitz constants ;
- For any $m\geq 1$ all series $\sum_n \frac{1}{w_1(\lambda)\cdots w_n(\lambda)^{1/m}} e_n$ are convergent.

Here we use the Lemma with $m=\min\mathcal{I}$.

Common Hypercyclic Algebras - The coordinatewise product

Corollary

Let X a Fréchet sequence space with an unconditional basis (e_n). Let $w = (w_n)$ a weighted sequence inducing a bounded operator on X and define

$$\lambda_w := \inf \left\{ \lambda > 0 : \text{for all } m > 0, \sum_n \frac{1}{\lambda^{\frac{n}{m}} (w_1 \cdots w_n)^{\frac{1}{m}}} e_n \text{ converges} \right\}.$$

Then $\bigcap_{\lambda > \lambda_w} HC(\lambda B_w) \cup \{0\}$ contains a non-trivial algebra.

These results apply not only to the classical families $(\lambda D)_{\lambda > 0}$ on $H(\mathbb{C})$ or $(\lambda B)_{\lambda > 1}$ on c_0 or ℓ_p but also for non-classical ones like $(B_{w(\lambda)})_{\lambda > 0}$ on c_0, where $w_n(\lambda) := 1 + \frac{\lambda}{n}, \lambda > 0$.

Common Hypercyclic Algebras - The Cauchy product

Theorem (using the Lemma with $m = \max \mathcal{I}$)

Let $\Lambda \subset \mathbb{R}$ be an interval, let X be a regular Fréchet sequence algebra under the Cauchy product and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weighted sequences, such that all functions $\log(w_n)$ are non-decreasing and Lipschitz on compact sets with uniformly bounded Lipschitz constants. Then $(T_\lambda)_{\lambda \in \Lambda}$ admits a common hypercyclic algebra if

- for all $\gamma \in \Lambda$,
$$\sum_{n=1}^{\infty} \frac{1}{w_1(\gamma) \cdots w_n(\gamma)} e_n \in X;$$
- for all $m \in \mathbb{N}$ and all $[a_0, b_0] \subset \Lambda$ there exist $c \in (0,1)$ and $\kappa_0 > 1$ such that
$$\lim_{\substack{\sigma \to \infty \\ c\sigma \in \mathbb{N}}} \sum_{n=c\sigma}^{\sigma} \frac{[w_1(\kappa_0 a) \cdots w_{m\sigma}(\kappa_0 a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)\sigma+n}(a)} e_n = 0, \text{ for all } a \in [a_0, b_0].$$

Question : Can we apply these results about common hypercyclic algebras for families of non-shift-like operators ?

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Thank you !