Convolution operators supporting hypercyclic algebras

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(joint work with F. Bayart and D. Papathanasiou)

Séminaire d'analyse fonctionnelle de Lille

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Linear Dynamics

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- The underlying space we generally work with is an F-space, often a Fréchet space or even a Banach space.
- **Definition.** A *Fréchet space* is a vector space X endowed with a separating increasing sequence $(\|\cdot\|_n)_{n\geq 0}$ of seminorms which is complete in the metric given by

$$d(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \min(1, \|x-y\|_n).$$

Linear Dynamics

- In this presentation we are specially interested on the space

$$H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} : f \text{ is holomorphic}\},$$

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- A very important linear operator defined on this space is that of complex derivation:

$$\begin{aligned} D : H(\mathbb{C}) &\to H(\mathbb{C}) \\ f &\mapsto f' \end{aligned}$$

Hypercyclicity

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- **Definition.** We say that an operator $T : X \to X$ is *topologically transitive* if, for all couple of non-empty open sets (U, V) of X, there is $u \in U$ and $N \in \mathbb{N}$ such that $T^N(u) \in V$.

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Any vector in this set is hypercyclic for T. □

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$$u =$$

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Hypercyclic algebras

- **Definition.** We call a *Fréchet algebra* every Fréchet space X on which it is defined a product $\cdot : X \times X \to X$ satisfying, for all $x, y \in X$ and all $q \geq 0$,

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- First positive result in 2009, independently by Shkarin [10] and by Bayart and Matheron [4]: the operator of complex derivation D on $H(\mathbb{C})$ admit a hypercyclic algebra.

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Theorem

Let T be a continuous linear operator on the separable Fréchet algebra X. Suppose that, for all $1 \leq m_0 \leq m_1$ and all $U, V, W \subset X$ open and non-empty, with $0 \in W$, one can choose $m \in [\![m_0, m_1]\!]$ and find $u \in U$ and $N \in \mathbb{N}$ such that

$$\begin{cases} T^N(u^m) \in V \\ T^N(u^n) \in W, \quad \text{for } n = [\![m_0, m_1]\!] \backslash \{m\}. \end{cases}$$

Then T has a hypercyclic algebra.

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The idea is to fix polynomials $A \in U$, $B \in V$ and define the candidate u again "by blocks" as $u = A + R_N$, where R_N is a small perturbation with good properties:

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- $D^N(R_N^{m_1}) = B \in V$;
- for all $m_0 \leq n < m_1$, $\deg(u^n) < N$.

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- ϕ multiple of an exponential $\Longrightarrow \phi(D)$ is just a translation
  $\Longrightarrow \phi$ has no hypercyclic algebra.

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- 2021, Bayart, Papathanasiou, FCJr [3] $\Longrightarrow |\phi(0)| > 1$ (+ conditions)

Some useful properties

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- Each function $E(\lambda)$ is an eigenvector of $\phi(D)$ corresponding to the eigenvalue $\phi(\lambda)$, that is, $\phi(D)E(\lambda) = \phi(\lambda)E(\lambda)$.
- If $\Lambda \subset \mathbb{C}$ has an accumulation point, then $\text{span}\{E(\lambda) : \lambda \in \Lambda\}$ is dense in $H(\mathbb{C})$.

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Being fixed U, V, W open and non-empty in $H(\mathbb{C})$, we use the fact that $\text{span}\{E(\lambda) : \lambda \in \Lambda\}$ is dense in $H(\mathbb{C})$ whenever Λ has an accumulation points in order to find $A \in U$ and $B \in V$ (not polynomials but) combinations of the form

$$A = \sum_{l=1}^{p} a_l E(\gamma_l) \qquad \text{and} \qquad B = \sum_{j=1}^{q} b_j E(\lambda_j).$$

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The candidate is then defined in the form

$$u = \sum_{l=1}^{p} a_l E(\gamma_l) + \sum_{j=1}^{q} c_j E(z_j),$$

where we need to find $c_j, z_j, j = 1, ..., q$ that allow us to distinguish the main parcel in the main power from all the other terms.

How's the candidate defined now?

If we consider the main power of this candidate, we find something like

$$u^n = \sum_{d=0}^{n} \sum_{\substack{\mathbf{l} \in I_p^{n-d} \\ \mathbf{j} \in I_q^d}} \alpha(\mathbf{l}, \mathbf{j}, d, n) a_{\mathbf{l}} c_{\mathbf{j}} E\left(\gamma_{l_1} + \cdots + \gamma_{l_{n-d}} + z_{j_1} + \cdots + z_{j_d}\right).$$

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After applying $\phi(D)$ we get terms containing

$$c_{\mathbf{j}} \phi\left(\gamma_{l_1} + \cdots + \gamma_{l_{n-d}} + z_{j_1} + \cdots + z_{j_d}\right) E\left(\gamma_{l_1} + \cdots + \gamma_{l_{n-d}} + z_{j_1} + \cdots + z_{j_d}\right).$$

In the proof we consider $\gamma_1, ..., \gamma_p \in B(a,\delta)$ and $z_1, ..., z_q \in B(b,\delta)$.

Main theorem

Theorem

Let ϕ be an entire function of exponential type satisfying the following.

(a) *ϕ is not a multiple of an exponential function;*

(b) *for all $1 \leq m_0 \leq m_1$, there exist $m \in [\![m_0, m_1]\!]$ and $a, b \in \mathbb{C}$ such that*

  (i) $|\phi(mb)| > 1$

  (ii) *for all $n \in [\![m_0, m_1]\!]$ and all $d \in \{0, ..., n\}$ with $(n, d) \neq (m, m)$,*

$$|\phi(db + (n-d)a)| < 1.$$

Then $\phi(D)$ supports a hypercyclic algebra.

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$$|\phi(db + (n-d)a)| < |\phi(mb)|^{d/m}.$$

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Applications

Theorem (Bayart, 2019 [2])

Let ϕ be an entire function of exponential type such that $|\phi(0)| < 1$. Then $\phi(D)$ has a hypercyclic algebra if and only if ϕ is not multiple of an exponential.

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Theorem

Let ϕ be an entire function of exponential type such that $|\phi(0)| < 1$. If ϕ is not multiple of an exponential and there exists a direction w such that $|\phi(tw)| \leq 1$ for all $t \gg 0$, then $\phi(D)$ has a hypercyclic algebra.

Applications

Theorem (Bayart, 2019 [2])

Let ϕ be an entire function of exponential type such that $|\phi(0)| < 1$. Then $\phi(D)$ has a hypercyclic algebra if and only if ϕ is not multiple of an exponential.

Theorem

Let ϕ be an entire function of exponential type such that $|\phi(0)| > 1$. If ϕ is not multiple of an exponential and there exists a direction w such that $|\phi(tw)| \to 0$ as $t \to +\infty$, then $\phi(D)$ has a hypercyclic algebra.

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Theorem (Bès, Ernst, Prieto, 2020 [7])

Let ϕ be an entire function of exponential type such that $|\phi(0)| = 1$. If ϕ is of subesponential growth then $\phi(D)$ has a hypercyclic algebra.

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Theorem (Aron et al., 2007 [1])

Let p be a positive integer and let $f \in H(\mathbb{C})$. Also, let T be a non-trivial translation operator on $H(\mathbb{C})$. If a non-constant function $g \in H(\mathbb{C})$ belongs to $\mathrm{orb}(f^p, T)$ then the order of each zero of g is a multiple of p.

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Theorem (Bayart, 2019 [2])

Let $\phi \in H(\mathbb{C})$ be an entire function of exponential type which is not multiple of an exponential. Then there exists a residual set of functions $f \in H(\mathbb{C})$ such that, for all $m \geq 1$, f^m is hypercyclic for $\phi(D)$.

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- Could we adapt the same ideas for operators of the form $\lambda I + B_w$?
- On the contrary, could we find a method to unprove the existence of hypercyclic algebras for operators of the form $\lambda I + B_w$?

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Let $\phi \in H(\mathbb{C})$ be an entire function of exponential type which is not multiple of an exponential. Suppose that, for all $\rho \in (0,1)$, there exists $w_0 \in \mathbb{C}$ such that

(i) $|\phi(w_0)| > 1$;

(ii) *for all* $r \in (0, \rho]$, $|\phi(rw_0)| < 1$.

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- Can we get a similar result for the case $|\phi(0)| > 1$ and the exists a direction $w \in \mathbb{C}$ such that $|\phi(tw)| \to 0$ as $t \to +\infty$?
- Is the condition of the main theorem enough?

One last problem

We know that convolution operators admit upper frequent hypercyclic subspaces, so the search for upper frequent hypercyclic algebras is a natural step forward.

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- Could we get upper frequent hypercyclic algebras for convolution operators?

A transitivity-like key result providing these algebras already exists!

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Merci de votre attention !