Self-similar fractals and common hypercyclicity

45th Summer Symposium in Real Analysis

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Common hypercyclicity though a Baire argument

Theorem (Basic Criterion)

Let Λ be a compact topological space and let $(T_\lambda)_{\lambda\in\Lambda}$ be a continuous family of operators acing on the same Banach space X. Suppose that, for every pair $(u,v)\in\mathcal{D}\times\mathcal{D}$, every $\varepsilon>0$, there exist $\lambda_1,\dots,\lambda_q\in\Lambda$, sets $\Lambda_1,\cdots,\Lambda_q$ with $\lambda_i\in\Lambda_i$ and positive integers $n_1,\dots,n_q$ such that

1. $\bigcup_i \Lambda_i \supset \Lambda$,
2. $\|\sum_i S_{\lambda_i}^{n_i} v\| \leq \varepsilon$,
3. *for all* $i=1,...,q$ *and all* $\lambda\in\Lambda_i$, $\|\sum_{j\neq i} T_\lambda^{n_i} S_{\lambda_j}^{n_j} v\| \leq \varepsilon$,
4. *for all* $i=1,...,q$ *and all* $\lambda\in\Lambda_i$, $\|T_\lambda^{n_i} u\| \leq \varepsilon$
5. *for all* $i=1,...,q$ *and all* $\lambda\in\Lambda_i$, $\|T_\lambda^{n_i} S_{\lambda_i}^{n_i} v - v\| \leq \varepsilon$.

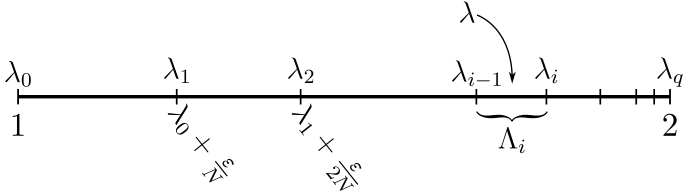
Then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is a dense G_δ subset of X.

Let us consider again the family of weights $w(a) = \big(w(a)\big)_{a\in[1,2]}$ given by $w_1(a)\cdots w_n(a) = \exp(an)$, $T_a := B_{w(a)}$ and $S_a := F_{w(a)^{-1}}$. For the property (5) of the Basic Criterion, we get

$$\|T_\lambda^{n_i} S_{\lambda_i}^{n_i} v - v\| \lesssim n_i|\lambda - \lambda_i|.$$

Thus, the covering of $[1,2]$ has to be made of intervals $\Lambda_1,\ldots,\Lambda_q$ with $\operatorname{diam}(\Lambda_i) \le \frac{\varepsilon}{n_i}$.

We can take $n_i = iN$ and define a partition of the form



⋙ Note that $n_i = iN$ is the absolute best that we can do! ⋘

General criterion in one dimension

Theorem (Costakis and Sambarino (2004))

Assume that, for each $v \in \mathcal{D}$ and each compact interval $K \subset \Lambda$, the following properties hold true, where $\lambda, \mu \in K$.

(1) *There exist $\kappa \in \mathbb{N}$ and a summable sequence of positive numbers $(c_k)_k$ such that*
   - $\|T_\lambda^{n+k} S_\mu^n(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\mu \leq \lambda$;
   - $\|T_\lambda^{n} S_\mu^{n+k}(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\lambda \leq \mu$.

(2) *Given $\eta > 0$, one can find $\tau > 0$ such that, for all $n \geq 1$,*

$$0 \leq \mu - \lambda \leq \frac{\tau}{n} \implies \|T_\lambda^n S_\mu^n(v) - v\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

$\ggg$ We call this the *CS-Criterion.* $\lll$

We are interested in the 2-dimensional case. Let $\Lambda \subset \mathbb{R}_+^2$ compact and

$$T_\lambda = B_{w(x)} \times B_{w(y)}, \quad \forall \lambda = (x, y) \in \Lambda,$$

acting on $X = \ell_p(\mathbb{N}), p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$, where

$$w_1(a) \cdots w_n(a) = \exp(an^\alpha), \quad \alpha \in (0, 1].$$

In order to apply the basic criterion, we we want:

- to cover Λ, say by squares $\Lambda_1, \dots \Lambda_q$;
- to tag these squares, say by $\lambda_1, \dots, \lambda_q$;
- to associate them to powers $n_1 < \cdots < n_q$.

We write these tags in the form $\lambda_j = (x_j, y_j)$.

“Jump” constraints

When you try to verify condition (2) of the Basic Criterion, you fix $i \in \{1, \ldots, q\}, \lambda_i \in \Lambda_i$ and you find something like

$$\sum_{j \neq i} B^{n_i}_{w(x_i)} F^{n_j}_{w(x_j)^{-1}}(v) \approx \sum_{j=i+1}^{q} \frac{1}{\exp((x_j - x_i)n_j^\alpha + x_i(n_j - n_i)^\alpha)} e_{n_j - n_i},$$

and another symmetrical condition for the second coordinate. Hence, for this to be small, we need at least, for all $1 \leq i < j \leq q$,

$$(x_j - x_i)n_j^\alpha + x_i(n_j - n_i)^\alpha > 0 \quad \text{and} \quad (y_j - y_i)n_j^\alpha + y_i(n_j - n_i)^\alpha > 0.$$

This is a multi-dimensional difficulty. Either $x_j < x_i$ or $y_j < y_i$ with $j > i$ is called a “backward jump”. They cannot be avoided.

Note that we can rewrite these constraints into something like

$$\|\lambda_j - \lambda_i\|_\infty \leq D \frac{(n_j - n_i)^\alpha}{n_j^\alpha}, \quad \text{for some } D > 0.$$

Almost there

Theorem (Bayart, Menet, FCJr, 2022)

For any square $\Lambda \subset \mathbb{R}^2_+$ (and X, $(w_n(x))_{x>0}$ as before),

$$\alpha < \frac{1}{2} \implies \bigcap_{\lambda \in \Lambda} HC(B_{w(x)} \times B_{w(y)}) \neq \varnothing,$$

$$\alpha > \frac{1}{2} \implies \bigcap_{\lambda \in \Lambda} HC(B_{w(x)} \times B_{w(y)}) = \varnothing.$$

(This is a consequence of a generalized version of the CS-Criterion).

The limit case $\alpha = \frac{1}{2}$ corresponds to the weights

$$w_1(x) \cdots w_n(x) = \exp(x\sqrt{n}).$$

Ideal bi-dimensional CS-Criterion

Theorem (bi-dimensional CS-Criterion)

Assume that there is $D > 0$ such that, for each $u \in \mathcal{D}$ and each compact square K, the following properties hold true, where $\lambda, \mu \in K$.

(1) *There exist $\kappa \in \mathbb{N}$ and a summable sequence of positive numbers $(c_k)_k$ such that, whenever $\|\lambda - \mu\|_\infty \leq D\sqrt{\frac{k}{n+k}}$, $n, k \in \mathbb{N}, k \geq \kappa$,*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq c_k \qquad \textit{and} \qquad \left\|T_\lambda^n S_\mu^{n+k} u\right\| \leq c_k.$$

(2) *Given $\eta > 0$, there is $\tau > 0$ such that, for all $n \geq 1$,*

$$\|\lambda - \mu\| \leq \frac{\tau}{\sqrt{n}} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

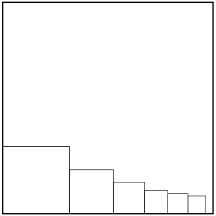
A covering game

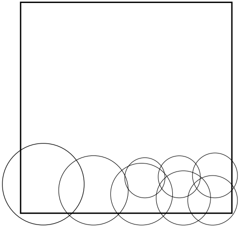
We would get a perfect generalization of the CS-Criterion if the sequence of powers $(n_i)_i$ could be taken as $n_i = iN$.

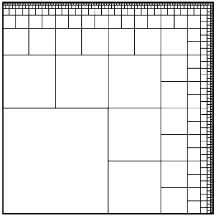
One can think of this as a covering game: we are given initial values $\tau > 0$ and $N \in \mathbb{N}$ and we have to conveniently cover our parameter set by squares of sides $\frac{\tau}{\sqrt{N}}, \frac{\tau}{\sqrt{2N}}, \frac{\tau}{\sqrt{3N}}, \ldots$ and so on.

How to organize these squares is the difficult part of the process.

There are multiple ways of organizing the pieces







We can also consider a canonical dyadic covering, that is, by 4^m squares of side $1/2^m$. There are many ways of ordering this covering.

Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}
Λ_{9}	Λ_{10}	Λ_{11}	Λ_{12}
Λ_{5}	Λ_{6}	Λ_{7}	Λ_{8}
Λ_{1}	Λ_{2}	Λ_{3}	Λ_{4}

(1) impossible

Λ_{16}	Λ_{15}	Λ_{14}	Λ_{13}
Λ_{9}	Λ_{10}	Λ_{11}	Λ_{12}
Λ_{8}	Λ_{7}	Λ_{6}	Λ_{5}
Λ_{1}	Λ_{2}	Λ_{3}	Λ_{4}

(2) impossible

Λ_{16}	Λ_{15}	Λ_{12}	Λ_{11}
Λ_{13}	Λ_{14}	Λ_{9}	Λ_{10}
Λ_{4}	Λ_{3}	Λ_{8}	Λ_{7}
Λ_{1}	Λ_{2}	Λ_{5}	Λ_{6}

(3) ok for $\alpha < 1/2$

Could dyadic partitions be enough for $\alpha = \frac{1}{2}$?

Figure : Dyadic partition of order 4

Suppose that we are allowed to take the best sequence $n_i = iN$.

For the dyadic partition of order m, we divide the square in 4^m sub-squares of side $\frac{1}{2^m}$. Our covering has 4^m parts of the form

$$\Lambda_i = \left[x_i, x_i + \tfrac{\tau}{\sqrt{n_i}}\right] \times \left[y_i, y_i + \tfrac{\tau}{\sqrt{n_i}}\right],$$

for $i = 1, \ldots, 4^m$.

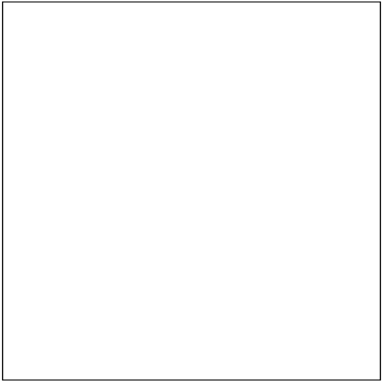
Since $n_i = iN$, the last part Λ_{4^m} is a square of side

$$\frac{\tau}{\sqrt{n_{4^m}}} = \frac{\tau}{\sqrt{4^m N}} = \frac{1}{2^m}\frac{\tau}{\sqrt{N}} \ll \frac{1}{2^m},$$

thus Λ_{4^m} will never cover a dyadic sub-square of order m, for $N \gg 0$.

A new approach

Let us explain the method for a simple case: $\frac{\tau}{\sqrt{N}} = \frac{1}{2}$. We aim to cover a unit square by square pieces of side $\frac{\tau}{\sqrt{N}}, \frac{\tau}{\sqrt{2N}}, \frac{\tau}{\sqrt{3N}}, \ldots$



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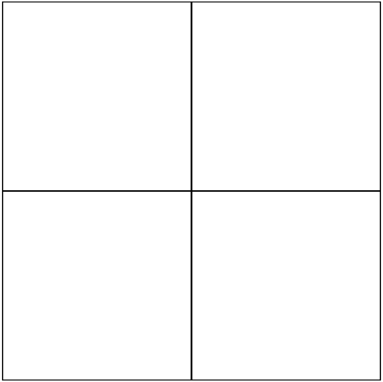

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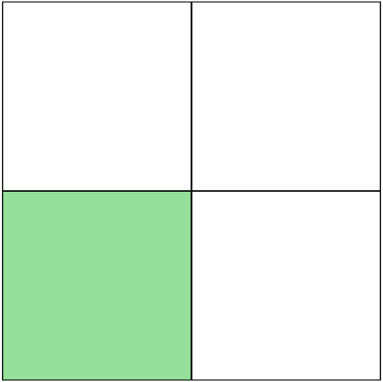

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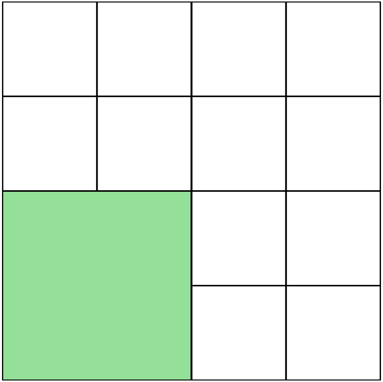

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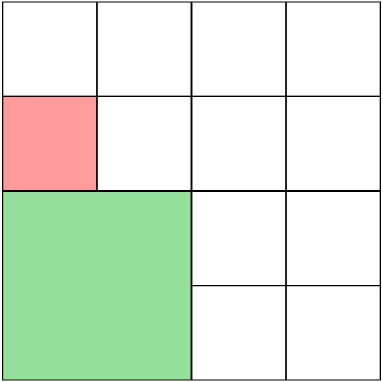

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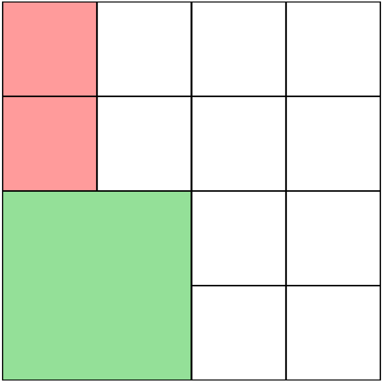

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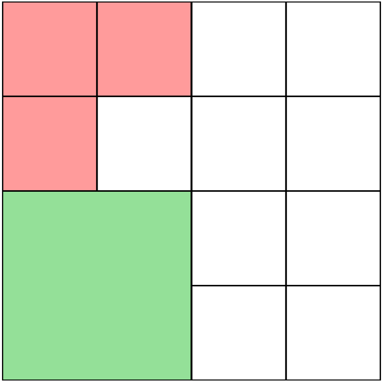

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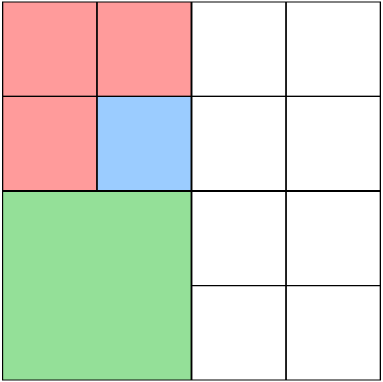

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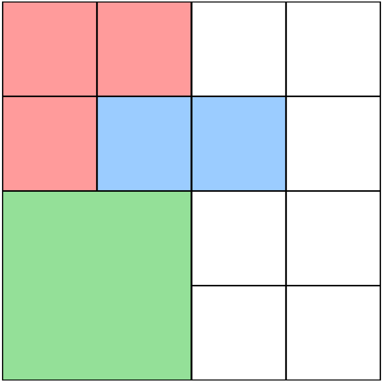

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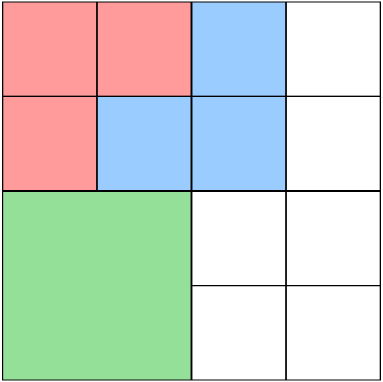

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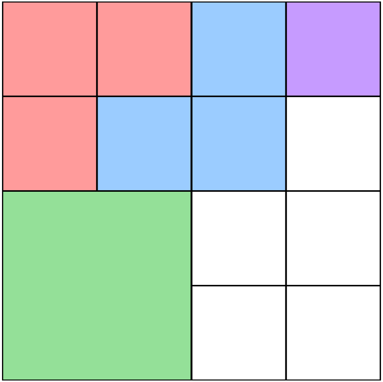

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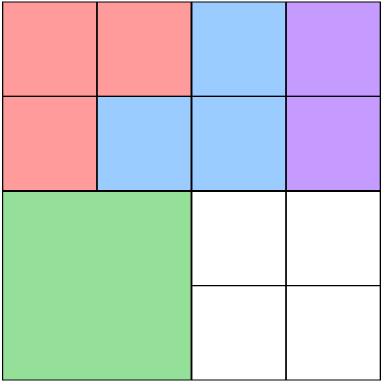

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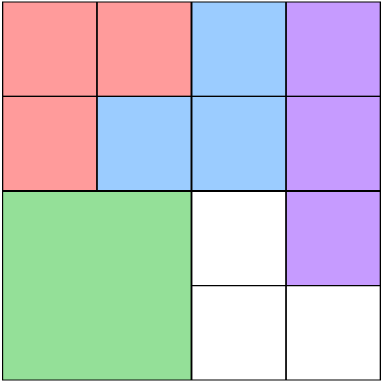

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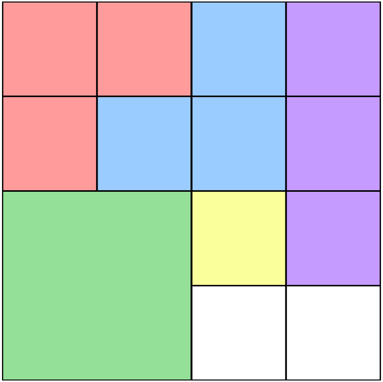

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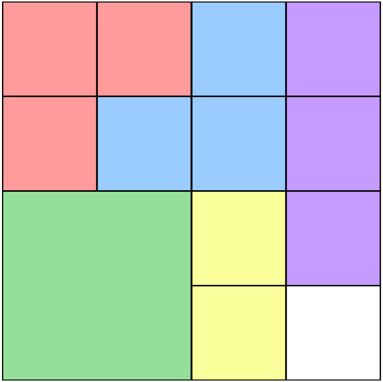

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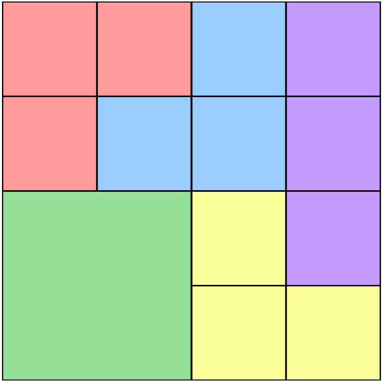

A new approach: using pseudo-Hilbert curves

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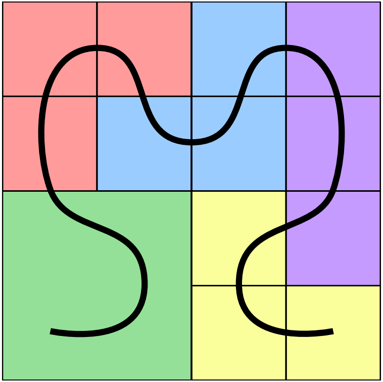
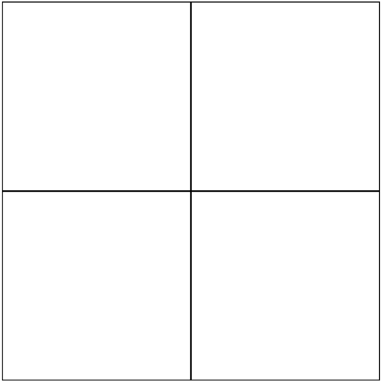


A covering game: round 1



Value used:

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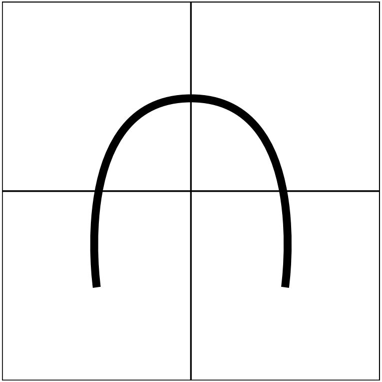

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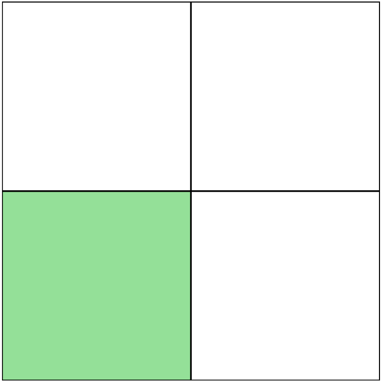

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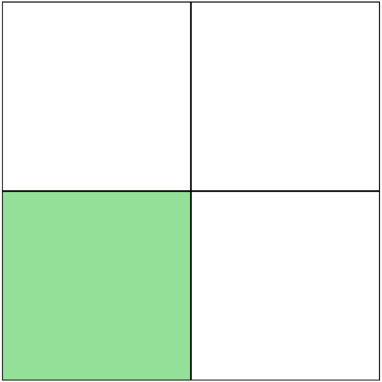

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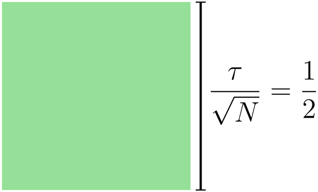
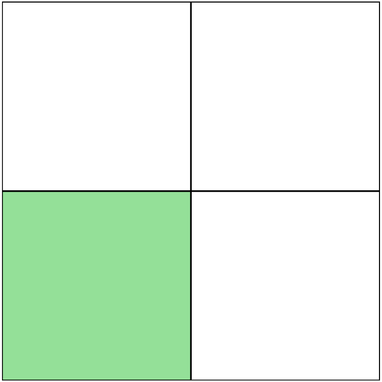


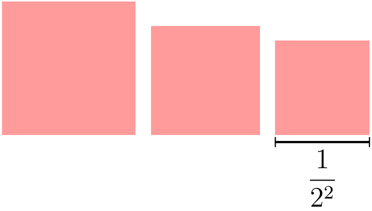
Round 1: victory!

A covering game: round 2

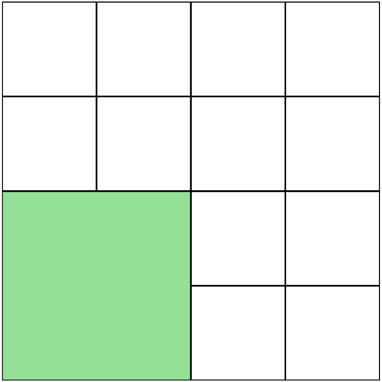


Values used:

$$\frac{\tau}{\sqrt{2N}} > \frac{\tau}{\sqrt{3N}} > \frac{\tau}{\sqrt{4N}} = \frac{1}{2^2}$$



A covering game: round 2



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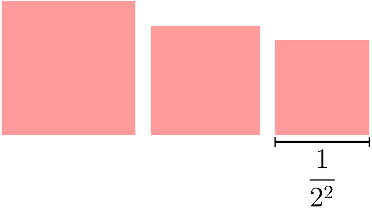

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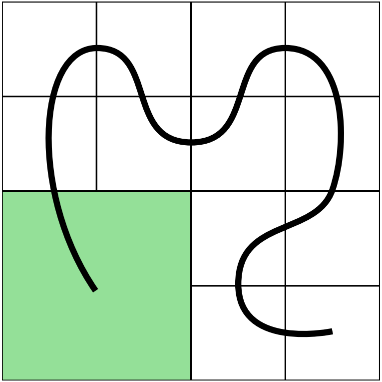

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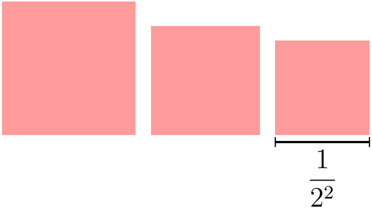

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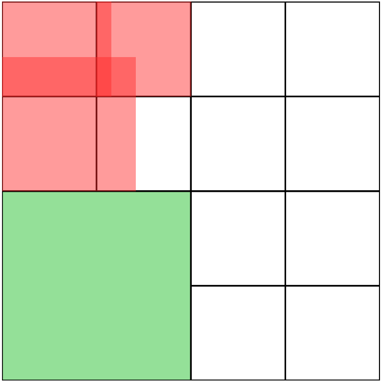

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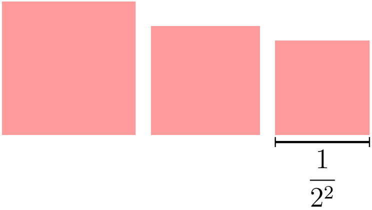
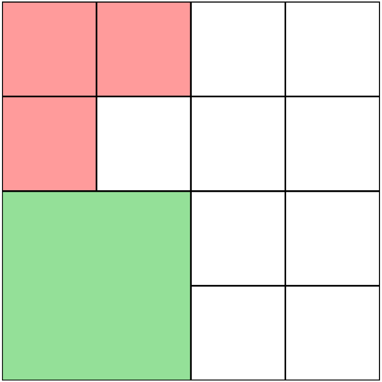


A covering game: round 2



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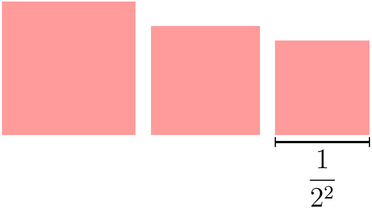

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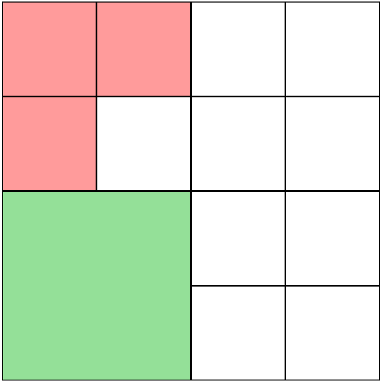

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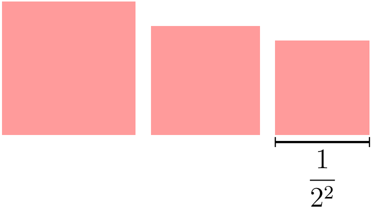
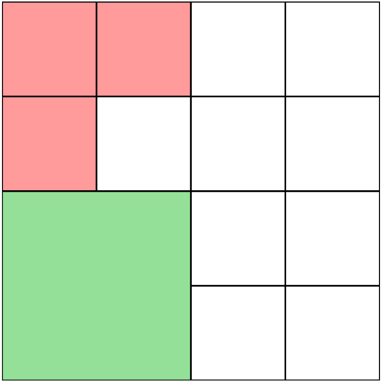


Round 2: victory!

A covering game: round 3



Values used:

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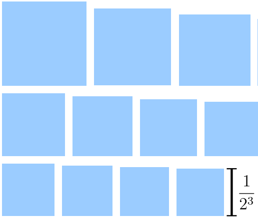

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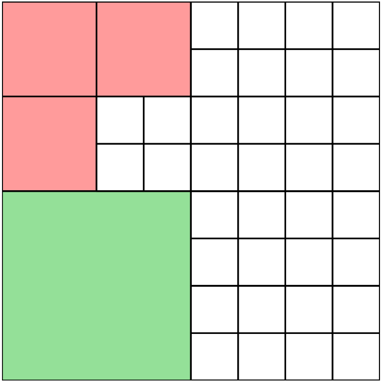

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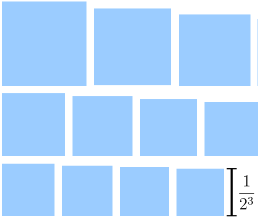

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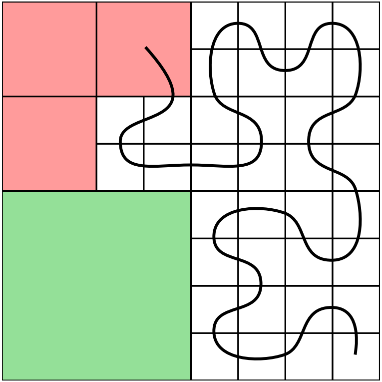

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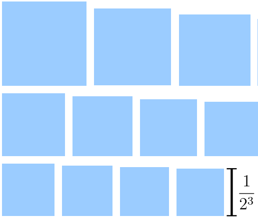

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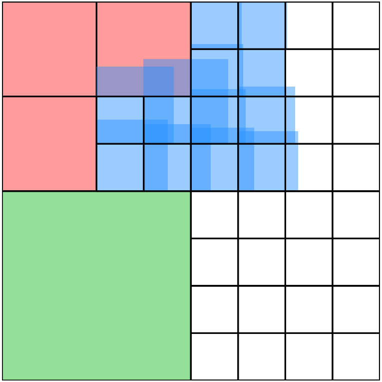

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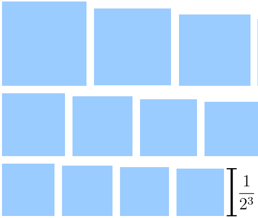

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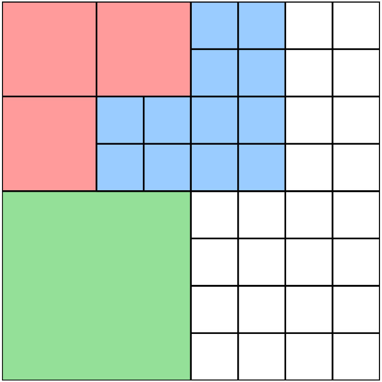

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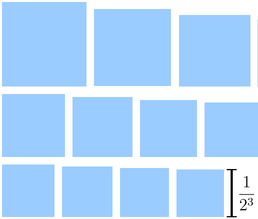

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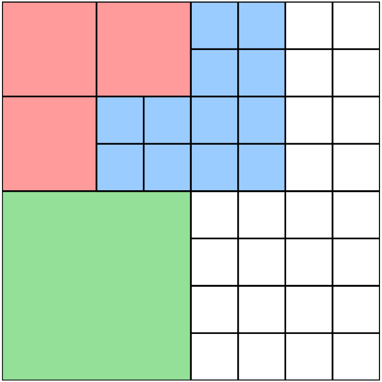

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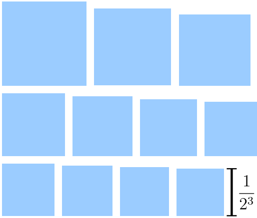
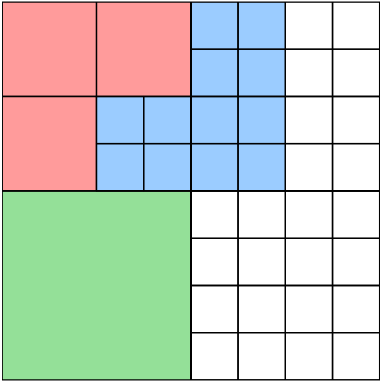


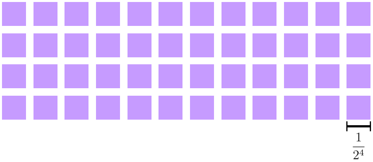
Round 3: victory!

A covering game: round 4

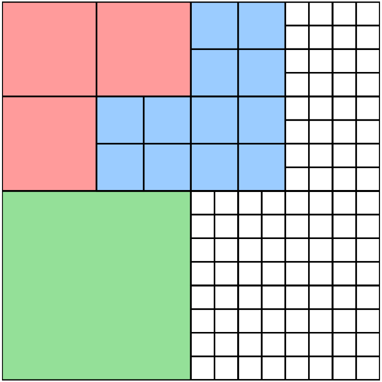


Values used:

$$\underbrace{\frac{\tau}{\sqrt{17N}} > \cdots > \frac{\tau}{\sqrt{64N}}}_{\text{to make } 48=3\times 4^2 \text{ pieces}} = \frac{1}{2^4}$$



A covering game: round 4



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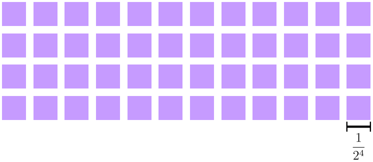
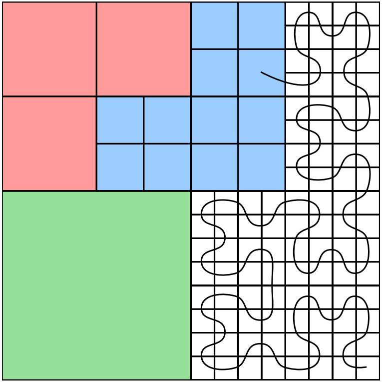


A covering game: round 4



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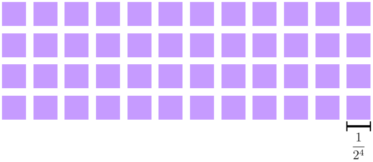

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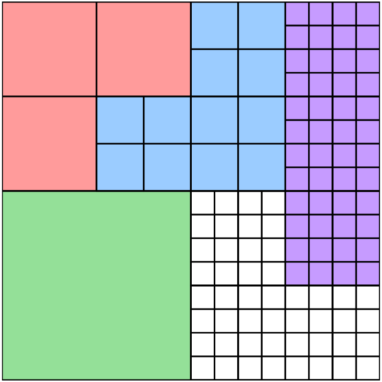

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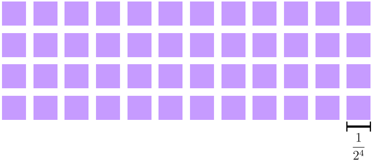

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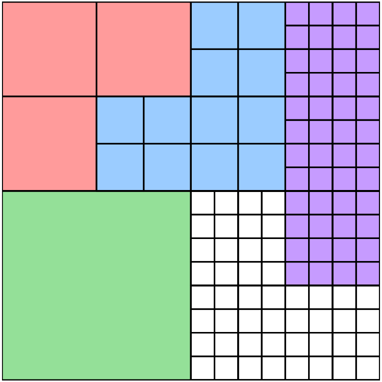

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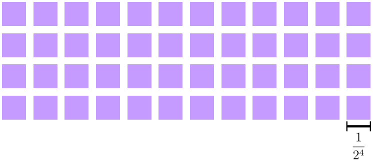
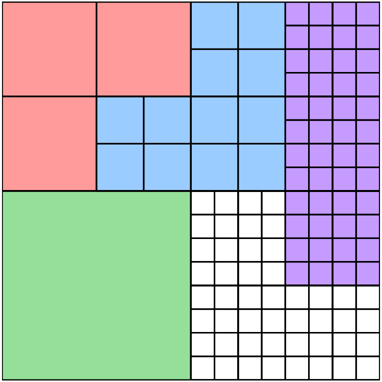


Round 4: victory!

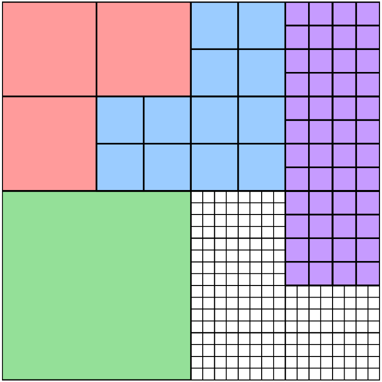
A covering game: round 5



Values used:

$$\underbrace{\frac{\tau}{\sqrt{65N}} > \cdots > \frac{\tau}{\sqrt{256N}}}_{\text{to make } 192=3\times 4^3 \text{ pieces}} = \frac{1}{2^5}$$

A covering game: round 5



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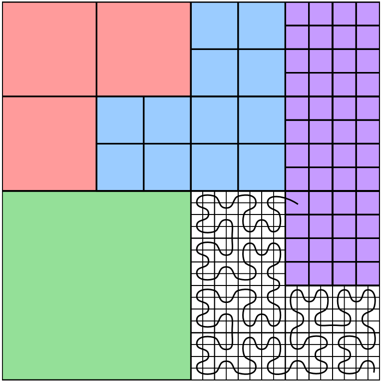

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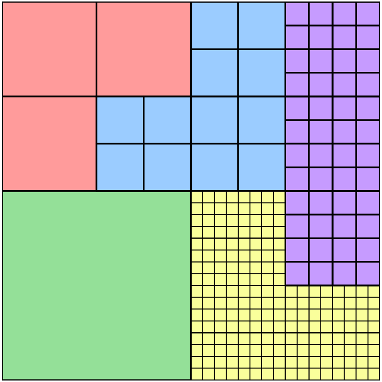

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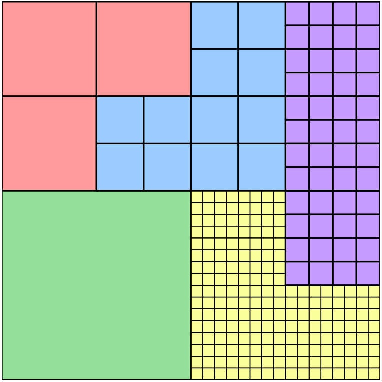

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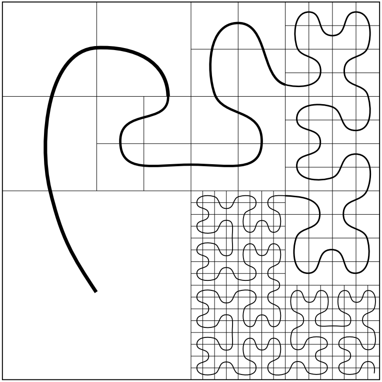
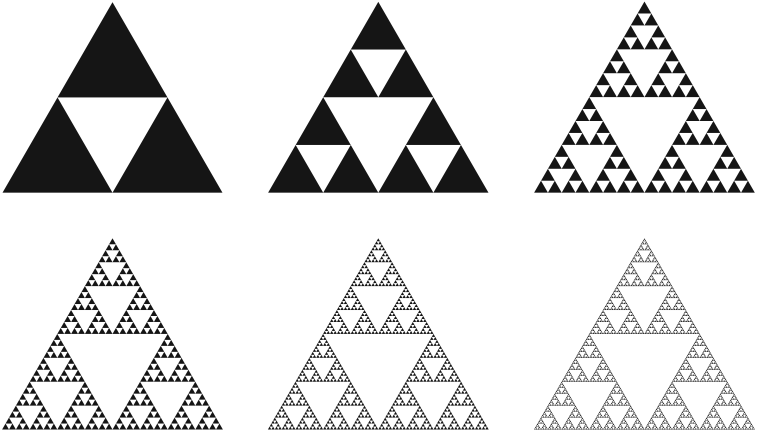
Round 5: victory. The game is over.

How the enumeration of our covering looks like



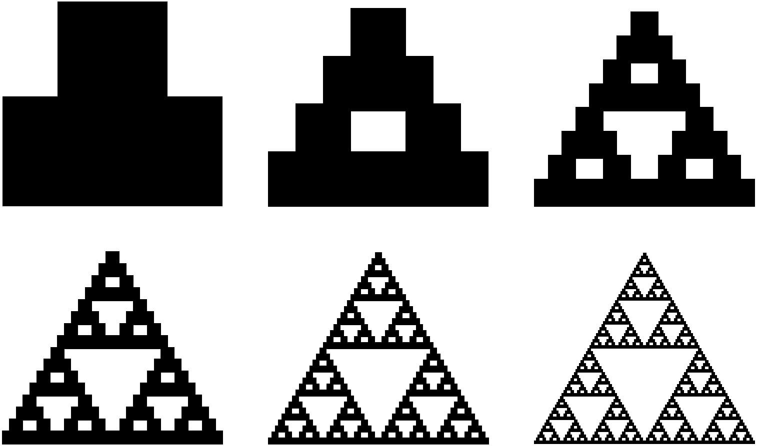
Ordering of $\Lambda_1, \Lambda_2, \ldots, \Lambda_{256}$ such that $K \subset \bigcup_{i=1}^{256} \Lambda_i$

Sierpiński gasket



Obtaining the Sierpiński gasket through triangles

Sierpiński gasket



Obtaining the Sierpiński gasket through squares

Dimension of the gasket

Let Γ be a unitary Sierpiński gasket inside $[0,+\infty)^2$. The similarity dimension (= Hausdorff dimension) of the gasket is $\dim_{\mathcal{H}} = \frac{\log 3}{\log 2}$. Up to now, we knew that the weights of the form

$$w_1(x)\cdots w_n(x) = \exp(xn^{\alpha})$$

induce a bi-dimensional family $\left(B_{w(x)} \times B_{w(y)}\right)_{(x,y)\in\Gamma}$ with a common hypercyclic vector whenever $\alpha < \frac{1}{\dim_{\mathcal{H}} \Gamma} = \frac{\log 2}{\log 3}$.

Applying the new approach to the gasket

The limit case we aim to address is $\alpha = \frac{1}{\frac{\log 3}{\log 2}} = \frac{\log 2}{\log 3}$.

Once again, we are given values $\tau > 0$, $N \in \mathbb{N}$ and we want to construct a covering by squares of sides

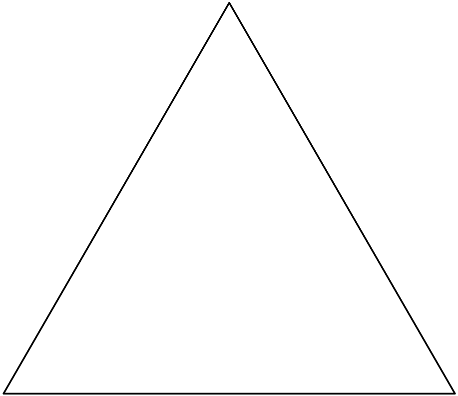
$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}}, \frac{\tau}{(2N)^{\frac{\log 2}{\log 3}}}, \frac{\tau}{(3N)^{\frac{\log 2}{\log 3}}}, \ldots$$

to make pieces for our covering.

The covering game of the gasket: grouping method

Again, let us consider the simple case

$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}} = \frac{1}{2}.$$



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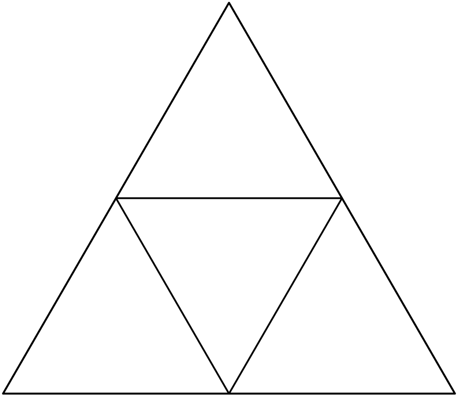

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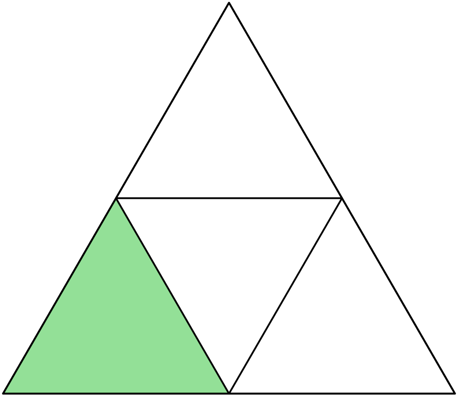

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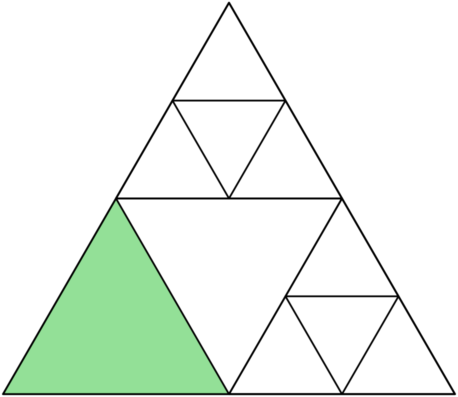

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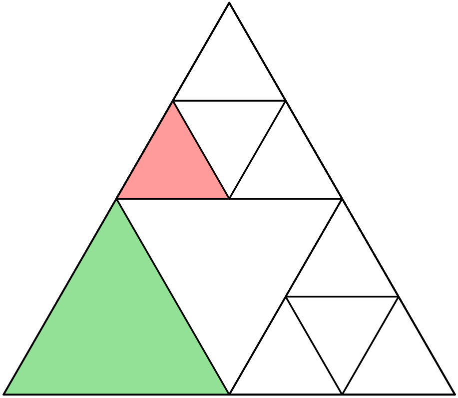

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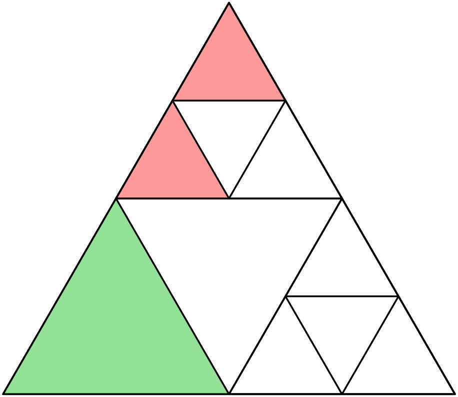

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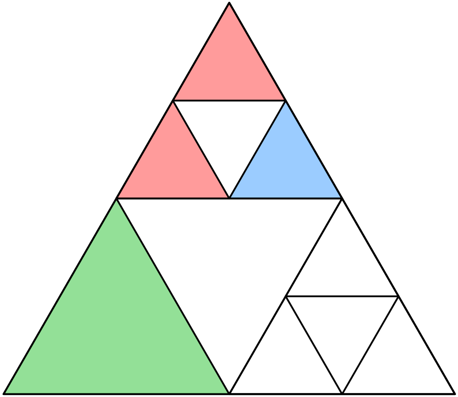

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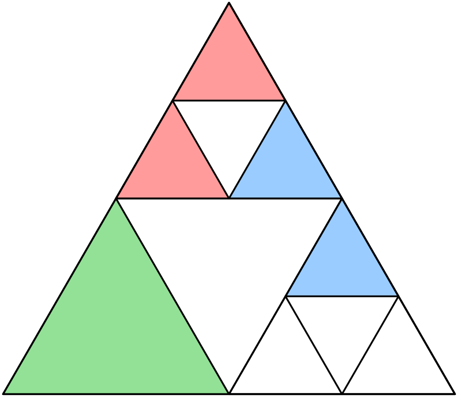

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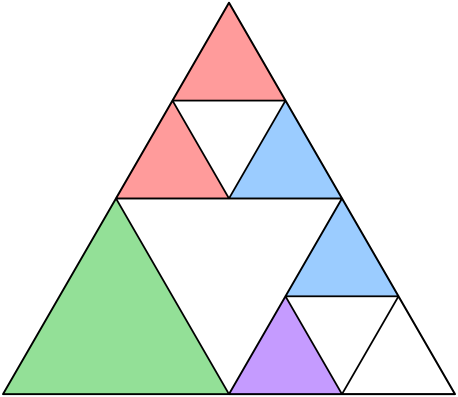

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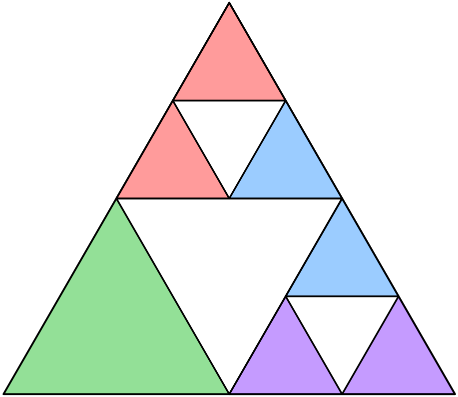

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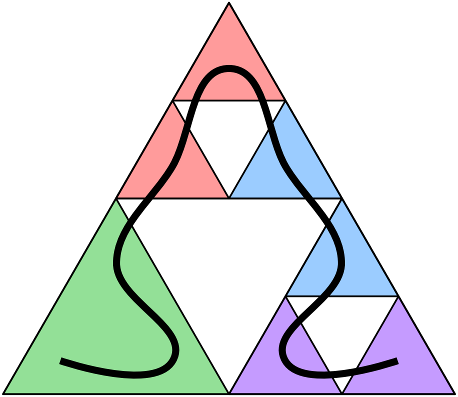
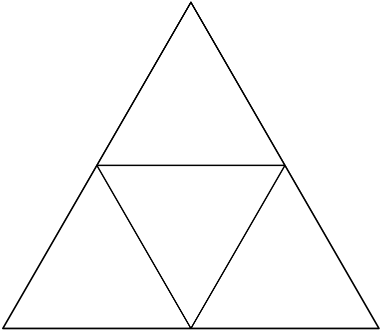


The covering game of the gasket: round 1



Value used:

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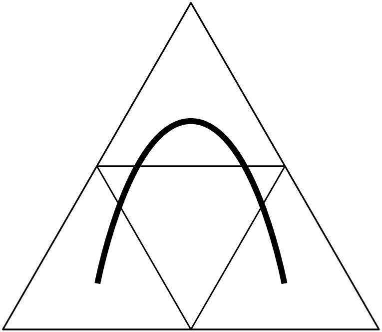
The covering game of the gasket: round 1



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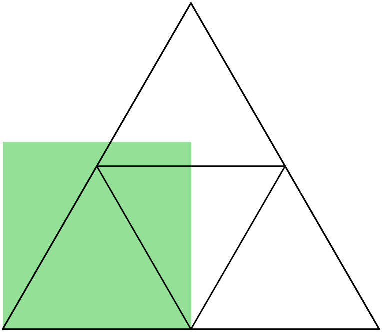
A covering game: round 1



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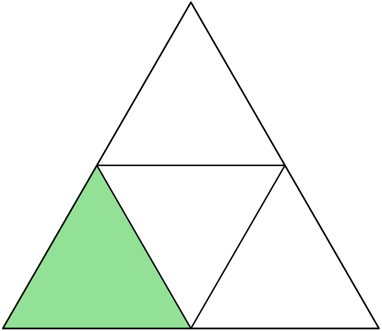

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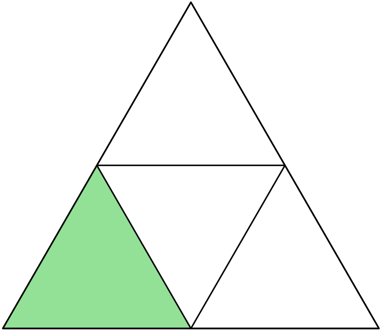

Value used:

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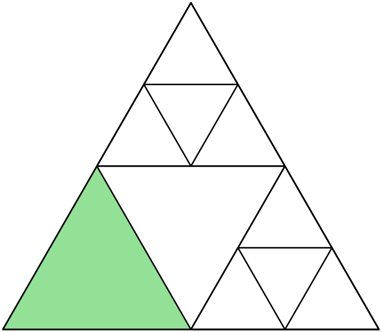
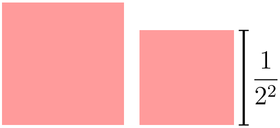
Round 1: victory!

The covering game of the gasket: round 2

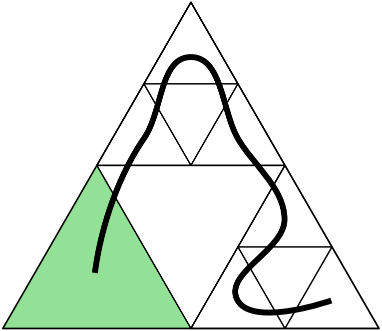


Values used:

$$\frac{\tau}{(2N)^{\frac{\log 2}{\log 3}}} > \frac{\tau}{(3N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^2}$$

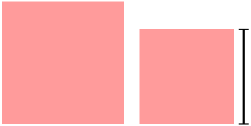


The covering game of the gasket: round 2



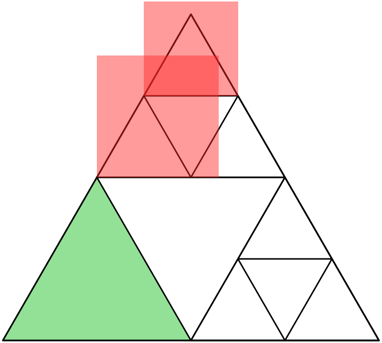
Values used:

$$\frac{\tau}{(2N)^{\frac{\log 2}{\log 3}}} > \frac{\tau}{(3N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^2}$$



$\frac{1}{2^2}$

A covering game: round 2



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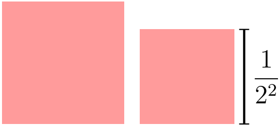
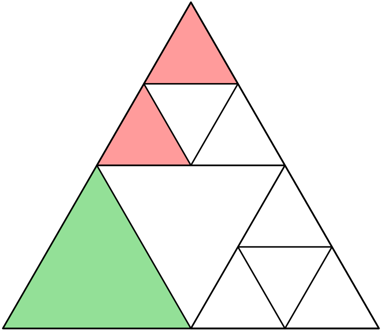


A covering game: round 2



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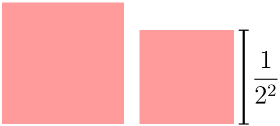
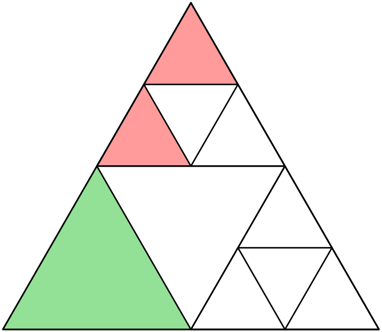


A covering game: round 2



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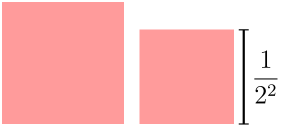
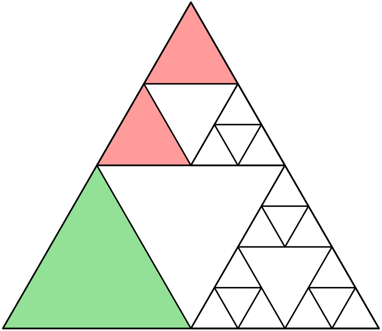


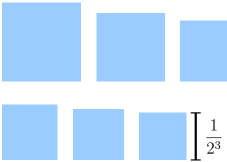
Round 2: victory!

The covering game of the gasket: round 3

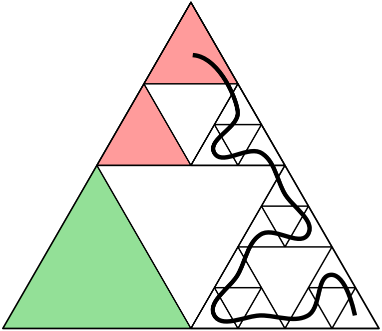


Values used:

$$\frac{\tau}{(4N)^{\frac{\log 2}{\log 3}}} > \cdots > \frac{\tau}{(9N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^3}$$



The covering game of the gasket: round 3



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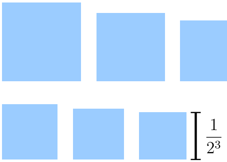
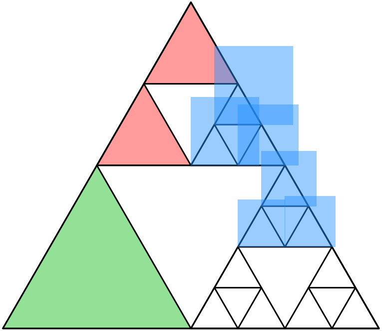


A covering game: round 3



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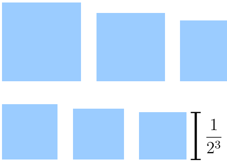

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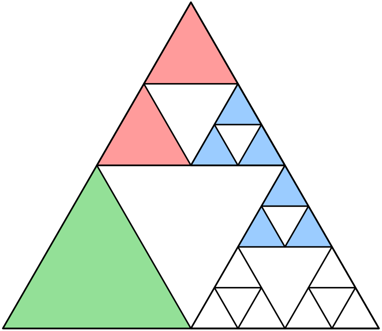

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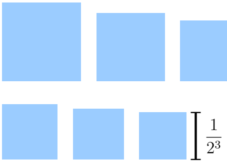
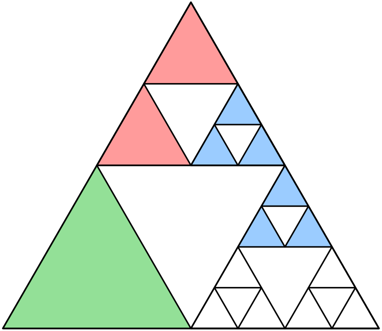


A covering game: round 3



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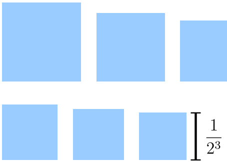
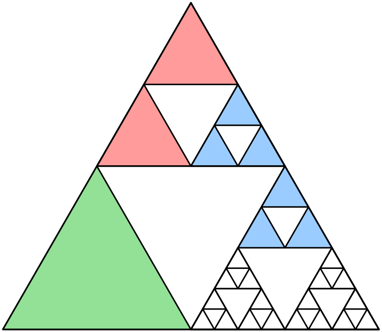


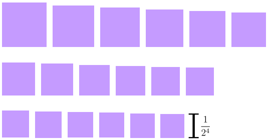
Round 3: victory!

The covering game of the gasket: round 4

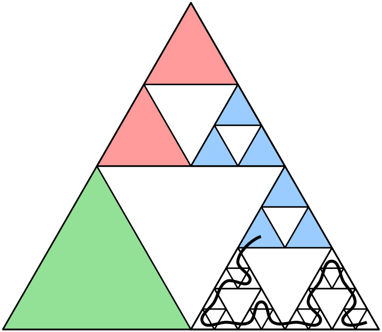


Values used:

$$\frac{\tau}{(10N)^{\frac{\log 2}{\log 3}}} > \cdots > \frac{\tau}{(27N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^3}$$



The covering game of the gasket: round 4



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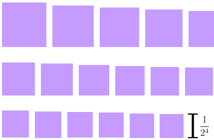
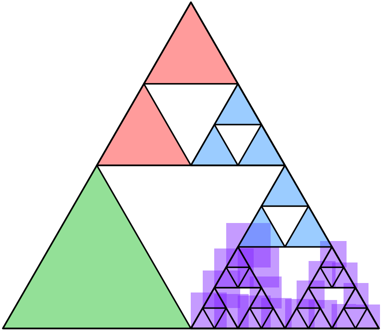


A covering game: round 4



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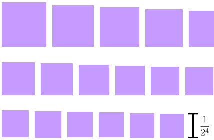
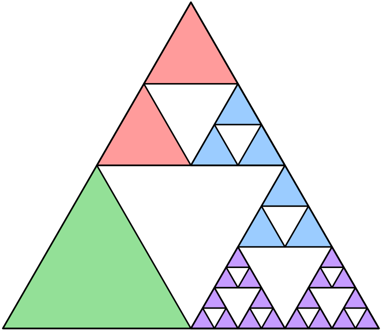


A covering game: round 4



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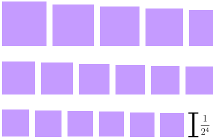
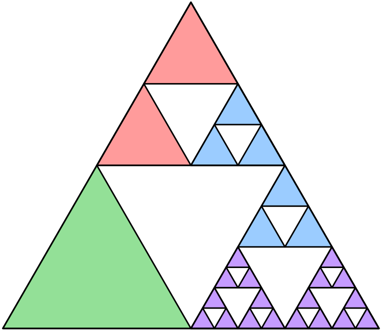


A covering game: round 4



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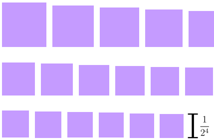
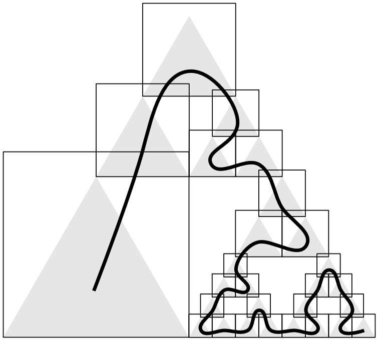


Round 4: victory! The game is over.

How the enumeration of our covering looks like



Ordering of $\Lambda_1, \Lambda_2, \ldots, \Lambda_{27}$ such that $\Gamma \subset \bigcup_{i=1}^{27} \Lambda_i$

Koch snowflake

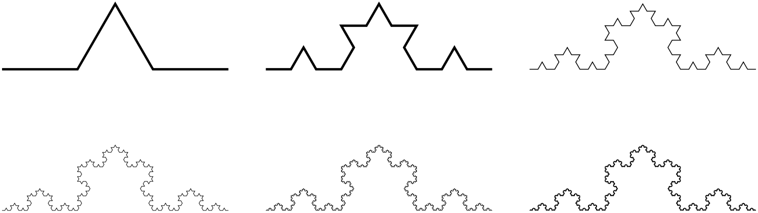


Figure : The von Koch curve forms the well known Koch snowflake. It has 4 similarities and contraction ration $\frac{1}{3}$. Its has Hausdorff dimension $\frac{\log(4)}{\log(3)}$.

We can apply our method by making groups of 3 patches of parts.

Minkowski sausage

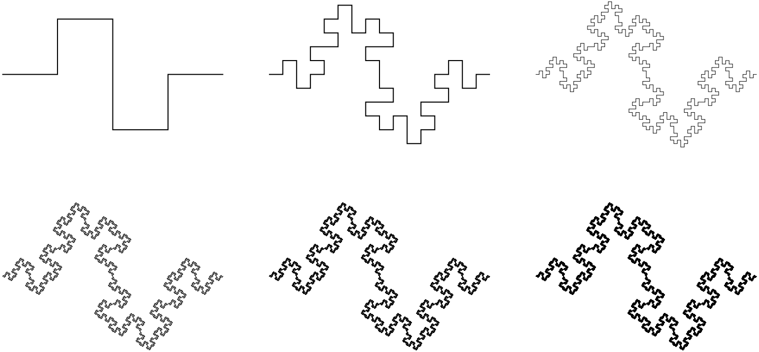


Figure : The Minkowski sasusage has 8 similarities with $\frac{1}{4}$ contraction ration. Its Hausdorff dimension is 1.5.

We can apply our method by making groups of 7 patches of parts.

A new concept of dimension?

Definition. Let $d \geq 1$ and $\Lambda \subset \mathbb{R}^d$ be compact. We say that Λ has *homogeneously ordered box dimension at most* $\gamma \in (0, d]$ if there exist $r \in [\![2, +\infty[\![$, $\rho > 0$ and, for all $m \geq 1$, a compact covering $(\Lambda_{\mathbf{i}})_{\mathbf{i} \in I_r^m}$ of Λ satisfying the following.

(i) For all $\mathbf{i} \in I_r^m$,

$$\mathrm{diam}(\Lambda_{\mathbf{i}}) \leq \rho \left(\frac{1}{r^{1/\gamma}} \right)^m .$$

(ii) For all $\mathbf{i} = (i_1, ..., i_m, i_{m+1}) \in I_r^{m+1}$,

$$\Lambda_{i_1, \ldots, i_{m+1}} \subset \Lambda_{i_1, \ldots, i_m} .$$

(iii) For all $\mathbf{i} = (i_1, ..., i_m) \in I_r^m$ and all $j \in \{2, ..., r\}$,

$$\Lambda_{\mathbf{i}, j-1, r} \cap \Lambda_{\mathbf{i}, j, 1} \neq \varnothing .$$

The *homogeneously ordered box dimension of* Λ (HBD$^\circ$ for short) is the smallest number γ such that Λ has homogeneously ordered box dimension at most γ. We will denote this number by $\dim_{HB}^\circ(\Lambda)$.

A full generalization of the CS-criterion

Theorem (Generalized Costakis-Sambarino Criterion)

Let $d \in \mathbb{N}$ and $\gamma > 0$ and let Λ be a compact subset of $\mathbb{R}^d$ having homogeneously ordered box dimension at most γ. Assume that there is $D > 0$ such that, for all $u \in \mathcal{D}$ and all compact $K \subset \Lambda$, the following properties hold true, where all parameters λ, μ belong to K.

(1) *There exist $\kappa \in \mathbb{N}$ and a sequence of positive numbers $(c_k)_{k\in\mathbb{N}}$ such that $\sum_{k=\kappa}^{\infty} c_k < \infty$ and, whenever $\|\lambda - \mu\|_\infty \leq D\frac{k^{1/\gamma}}{(n+k)^{1/\gamma}}$, with $n, k \in \mathbb{N}_0$ and $k \geq \kappa$, we have*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq c_k \qquad \text{and} \qquad \left\|T_\lambda^{n} S_\mu^{n+k} u\right\| \leq c_k.$$

(2) *Given $\eta > 0$, one can find $\tau > 0$ such that, for all $n \in \mathbb{N}$,*

$$\|\lambda - \mu\|_\infty \leq \frac{\tau}{n^{1/\gamma}} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

Applications

Let $d \geq 1$ and $\Gamma \subset \mathbb{R}^d$ compact. We write $\lambda = (x_1, \ldots, x_d)$ for any $\lambda \in \mathbb{R}^d$. Consider $\alpha \in (0, 1/d]$ and a family $\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big)_{\lambda \in \Gamma}$ induced by weights $\big(w(x)\big)_{x \in \mathbb{R}}$ satisfying

$$w_1(x) \cdots w_d(x) \approx \exp(xn^{\alpha}).$$

Theorem

If Γ has HBO$^\circ$ at most $\gamma \leq d$ and $0 < \alpha \leq 1/\gamma$, then

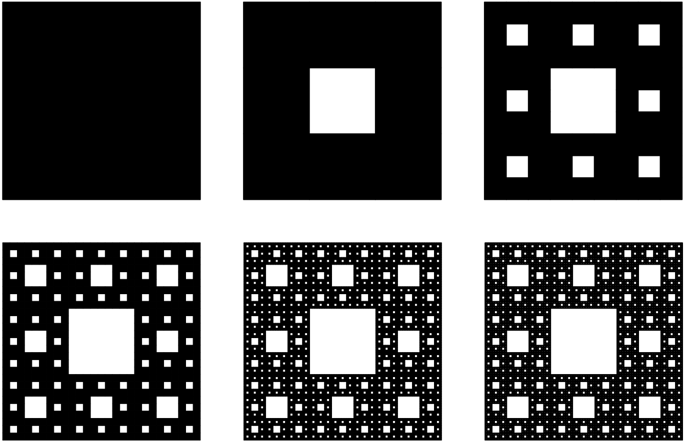
$$\bigcap_{\lambda \in \Gamma} HC\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big) \neq \varnothing.$$

Corollary

If Γ is a β-Hölder curve and $0 < \alpha \leq \beta$, then

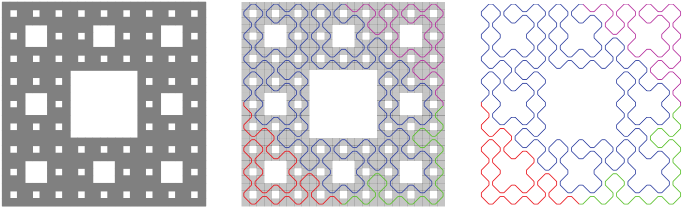
$$\bigcap_{\lambda \in \Gamma} HC\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big) \neq \varnothing.$$

IFS, GIFS and optimal parametrizations



Construction of the Sierpiński carpet.

IFS, GIFS and optimal parametrizations



Ordering the carpet (Rao and Zhang, 2016).

Some open questions

- $\sum \frac{1}{F(n)^2} = +\infty \implies \bigcap_{x,y>0} HC(B_{w(x)} \times B_{w(y)}) \neq \varnothing$?

- For $\mathcal{C} \subset \mathbb{R}_+^2$ the Cantor dust with dissection ratio $1/4$,

$$\bigcap_{(x,y)\in\mathcal{C}} e^x B \times e^y B \neq \varnothing?$$

- Can we, more generally, generalize our result for totally disconnected fractals?

- Can we adapt optimally our technique to self-similar fractals with non-uniform contraction ratio?

- For $\mathcal{T} \subset \mathbb{R}_+^2$ the 1-dimensional Takagi curve,

$$\bigcap_{(x,y)\in\mathcal{T}} e^x B \times e^y B \neq \varnothing?$$

Frame Title

Thanks for your attention!

References

[1] E. Abakumov and J. Gordon. **Common hypercyclic vectors for multiples of backward shift.** *Journal of Functional Analysis* 200.2 (2003): 494-504.

[2] G. Costakis and M. Sambarino. **Genericity of wild holomorphic functions and common hypercyclic vectors.** *Advances in Mathematics* 182.2 (2004): 278-306.

[3] F. Bayart and É. Matheron. **Dynamics of linear operators**. Cambridge Tracts in Mathematics No. 179. *Cambridge university press*, 2009.

[4] F. Bayart, F. Costa Jr. and Q. Menet. **Common hypercyclic vectors and dimension of the parameter set.** *Indiana University Mathematics Journal* 71.4 (2022): 1763-1795.

[5] A. Peris. **Common hypercyclic vectors for backward shifts**, *Operator Theory Seminar*, Michigan State University (2000/01).

[6] H. Rao and S.-Q. Zhang. **Space-filling curves of self-similar sets (I): iterated function systems with order structures.** *Nonlinearity* 29.7 (2016): 2112.