Self-similar fractals and common hypercyclicity

Anual meeting of the FNRS Group - Functional Analysis

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Hypercyclicity and common hypercyclicity

- Let X be a separable Banach space and $T : X \to X$ be linear and continuous (= operator).

We say that $x \in X$ is a **hypercyclic vector** for T when its orbit $Orb(x, T) := \{T^n x : n \geq 0\}$ is dense in X. If such a vector exists we say that T is a **hypercyclic operator** on X. The set of hypercyclic vectors for an operator T is denoted by $HC(T)$.

- Let $d \geq 1$, let $\Lambda \subset \mathbb{R}^d$ be compact and let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators acting on X such that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous (= continuous family).

We say that $x \in X$ is a **common hypercyclic vector** for $(T_\lambda)_{\lambda \in \Lambda}$ when $x \in HC(T_\lambda)$ for all $\lambda \in \Lambda$. If such a vector exists we say that $(T_\lambda)_{\lambda \in \Lambda}$ is a **common hypercyclic family** on X.

First common hypercyclicity results ($d = 1, 2$)

Theorem (Abakumov&Gordon (2003), Peris (2000/2001))

For $X = c_0(\mathbb{N})$ or $\ell_p(\mathbb{N}), 1 \leq p < \infty$, the family $(e^{\lambda}B)_{\lambda>0}$ has a common hypercyclic vector. (Even a G_δ-dense set of them!)

Borichev's example

For any $\Lambda \subset \mathbb{R}_+^2$ with positive Lebesgue measure, $(e^x B \times e^y B)_{(x,y)\in\Lambda}$ has no common hypercyclic vector on $\ell_1(\mathbb{N}) \times \ell_1(\mathbb{N})$.

These operators fall in the category of maps satisfying the following.

- There exists a dense subset $\mathcal{D} \subset X$ on which each T_λ has a right inverse $S_\lambda : \mathcal{D} \to X$.

First common hypercyclicity results ($d = 1$)

Theorem (Costakis and Sambarino (2004))

Assume that, for each $v \in \mathcal{D}$ and each compact interval $K \subset \Lambda$, the following properties hold true, where $\lambda, \mu \in K$.

(1) *There exist $\kappa \in \mathbb{N}$ and a summable sequence of positive numbers $(c_k)_k$ such that*
- $\|T_\lambda^{n+k} S_\mu^n(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\mu \leq \lambda$;
- $\|T_\lambda^{n} S_\mu^{n+k}(v)\| \leq c_k$ *whenever* $n \in \mathbb{N}, k \geq \kappa$ *and* $\lambda \leq \mu$.

(2) *Given $\eta > 0$, one can find $\tau > 0$ such that, for all $n \geq 1$,*

$$0 \leq \mu - \lambda \leq \tfrac{\tau}{n} \implies \|T_\lambda^n S_\mu^n(v) - v\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

$\ggg$ We call this the *CS-Criterion.* $\lll$

First common hypercyclicity results ($d \geq 1$)

Theorem (Basic criterion [3])

Assume the following. For each compact $K \subset \Lambda$, each pair $u, v \in \mathcal{D}$ and each open neighborhood O of the origing, one can find parameters $\lambda_1, ..., \lambda_q \in \Lambda$, sets of parameters $\Lambda_1, ..., \Lambda_q \subset \Lambda$, with $\lambda_i \in \Lambda_i, i = 1, ..., q$, and integers $n_1, ..., n_q \in \mathbb{N}$, such that

(i) $\displaystyle\bigcup_{i=1}^{q} \Lambda_i \supset K$;

(ii) $\forall i = 1, ..., q, \forall \lambda \in \Lambda_i : \displaystyle\sum_{j=1}^{q} S_{\lambda_j}^{n_j}(v) \in O$ *and* $\sum_{j \neq i} T_\lambda^{n_i} S_{\lambda_j}^{n_j}(v) \in O$;

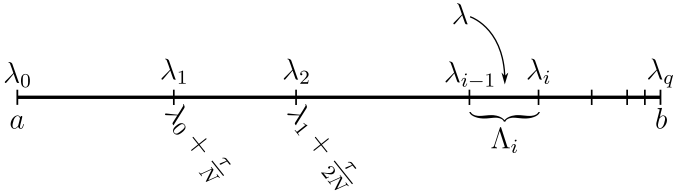
(iii) $\forall i = 1, ..., q, \forall \lambda \in \Lambda_i : T_\lambda^{n_i}(u) \in O$

(iv) $\forall i = 1, ..., q, \forall \lambda \in \Lambda_i : T_\lambda^{n_i} S_{\lambda_i}^{n_i}(u) - u \in O$.

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ *is a dense* G_δ *subset of* X.

Sketch of CS-Criterion $\implies$ Basic Criterion

Let $K = [a, b]$. Conditions (ii) and (iii) from the Basic Criterion come from condition (1) of CS-Criterion. Condition (iv) follows from (2) if we define $n_i = iN$ and discretize K by a partition of the form:



Since a partition is a covering, condition (i) follows and the proof is complete. ∎

⋙ Note that $n_i = iN$ is the absolute best we can do! ⋘

Families of products of weighted shifts

The family of hypercyclic Rolewicz operators $(e^{\lambda}B)_{\lambda>0}$ can be regarded as a family of weighted backward shifts $(B_{w(x)})_{x>0}$, where

$$w_1(x)w_2(x)\cdots w_n(x) = \underbrace{e^x e^x \cdots e^x}_{n \text{ times}} = \exp(xn).$$

Let us concentrate ourselves on families of weights of the form

$$w_1(x)w_2(x)\cdots w_n(x) = \exp(xn^{\alpha}), \quad \alpha \in (0,1].$$

We are interested in the 2-dimensional case. Let $\Lambda \subset \mathbb{R}^2_+$ compact and

$$T_\lambda = B_{w(x)} \times B_{w(y)}, \quad \forall \lambda = (x, y) \in \Lambda,$$

acting on $X = \ell_p(\mathbb{N}), p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$.

In order to apply the basic criterion, we we want:

- to cover Λ, say by squares $\Lambda_1, \dots \Lambda_q$;
- to tag these squares, say by $\lambda_1, \dots, \lambda_q$;
- to associate them to powers $n_1 < \cdots < n_q$.

We write these tags in the form $\lambda_j = (x_j, y_j)$.

“Jump” constraints

Verifying the basic criterion can be very tricky! When you try to verify condition (ii), you fix $i \in \{1, ..., q\}$ and you find

$$\sum_{j \neq i} B^{n_i}_{w(x_i)} F^{n_j}_{w(x_j)^{-1}}(v) \approx \sum_{j=i+1}^{q} \frac{1}{\exp((x_j - x_i)n_j^\alpha + x_i(n_j - n_i)^\alpha)} e_{n_j - n_i},$$

and another symmetrical condition for the second coordinate. Hence, for this to be small, we need at least, for all $1 \leq i < j \leq q$,

$$(x_j - x_i)n_j^\alpha + x_i(n_j - n_i)^\alpha > 0 \quad \text{and} \quad (y_j - y_i)n_j^\alpha + y_i(n_j - n_i)^\alpha > 0.$$

This is a multi-dimensional difficulty. Either $x_j < x_i$ or $y_j < y_i$ with $j > i$ is called a “backward jump”. They cannot be avoided.

Note that we can rewrite these constraints into something like

$$\|\lambda_j - \lambda_i\|_\infty \leq D \frac{(n_j - n_i)^\alpha}{n_j^\alpha}, \quad \text{for some } D > 0.$$

Almost there

Theorem (Bayart, Menet, FCJr, 2022)
For any square $\Lambda \subset \mathbb{R}^2_+$ *(and* X*,* $(w_n(x))_{x>0}$ *as before),*

$$\alpha < \frac{1}{2} \implies \bigcap_{\lambda \in \Lambda} HC(B_{w(x)} \times B_{w(y)}) \neq \varnothing,$$

$$\alpha > \frac{1}{2} \implies \bigcap_{\lambda \in \Lambda} HC(B_{w(x)} \times B_{w(y)}) = \varnothing.$$

(This is a consequence of a generalized version of the CS-Criterion).

The limit case $\alpha = \frac{1}{2}$ corresponds to the weights

$$w_1(x) \cdots w_n(x) = \exp(x\sqrt{n}).$$

Ideal bi-dimensional CS-Criterion

Theorem (bi-dimensional CS-Criterion)

Assume that there is $D > 0$ such that, for each $u \in \mathcal{D}$ and each compact square K, the following properties hold true, where $\lambda, \mu \in K$.

(1) *There exist $\kappa \in \mathbb{N}$ and a summable sequence of positive numbers $(c_k)_k$ such that, whenever $\|\lambda - \mu\|_\infty \leq D\sqrt{\frac{k}{n+k}}$, $n, k \in \mathbb{N}, k \geq \kappa$,*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq c_k \qquad \text{and} \qquad \left\|T_\lambda^n S_\mu^{n+k} u\right\| \leq c_k.$$

(2) *Given $\eta > 0$, there is $\tau > 0$ such that, for all $n \geq 1$,*

$$\|\lambda - \mu\| \leq \frac{\tau}{\sqrt{n}} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a G_δ-dense subset of X.

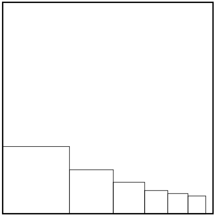
A covering game

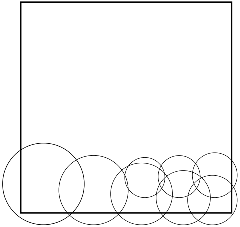
We would get a perfect generalization of the CS-Criterion if the sequence of powers $(n_i)_i$ could be taken as $n_i = iN$.

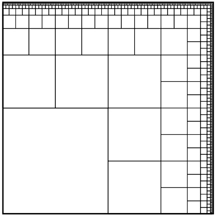
One can think of this as a covering game: we are given initial values $\tau > 0$ and $N \in \mathbb{N}$ and we have to conveniently cover our parameter set by squares of sides $\frac{\tau}{\sqrt{N}}, \frac{\tau}{\sqrt{2N}}, \frac{\tau}{\sqrt{3N}}, \ldots$ and so on.

How to organize these squares is the difficult part of the process.

There are multiple ways of organizing the pieces







We can also consider a canonical dyadic covering, that is, by 4^m squares of side $1/2^m$. There are many ways of ordering this covering.

Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}
Λ_{9}	Λ_{10}	Λ_{11}	Λ_{12}
Λ_{5}	Λ_{6}	Λ_{7}	Λ_{8}
Λ_{1}	Λ_{2}	Λ_{3}	Λ_{4}

(1) impossible

Λ_{16}	Λ_{15}	Λ_{14}	Λ_{13}
Λ_{9}	Λ_{10}	Λ_{11}	Λ_{12}
Λ_{8}	Λ_{7}	Λ_{6}	Λ_{5}
Λ_{1}	Λ_{2}	Λ_{3}	Λ_{4}

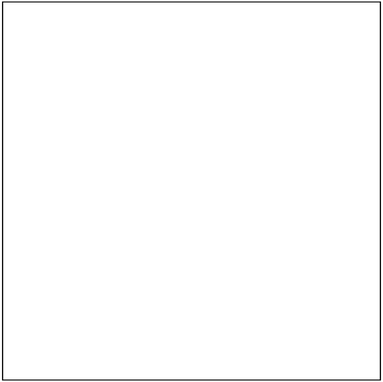
(2) impossible

Λ_{16}	Λ_{15}	Λ_{12}	Λ_{11}
Λ_{13}	Λ_{14}	Λ_{9}	Λ_{10}
Λ_{4}	Λ_{3}	Λ_{8}	Λ_{7}
Λ_{1}	Λ_{2}	Λ_{5}	Λ_{6}

(3) ok for $\alpha < 1/2$

A new approach

Let us explain the method for a simple case: $\frac{\tau}{\sqrt{N}} = \frac{1}{2}$. We aim to cover a unit square by square pieces of side $\frac{\tau}{\sqrt{N}}, \frac{\tau}{\sqrt{2N}}, \frac{\tau}{\sqrt{3N}}, \ldots$



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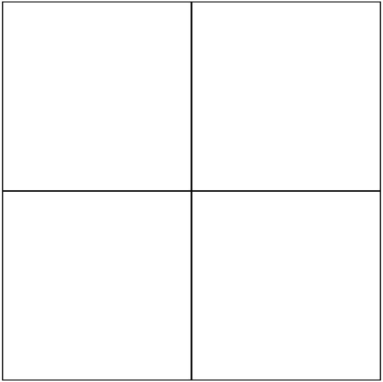

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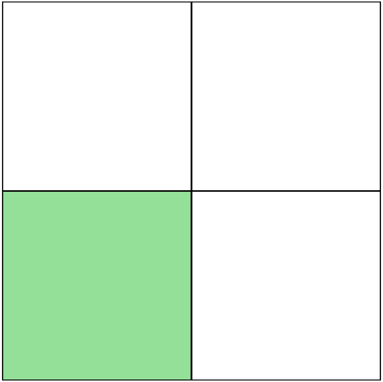

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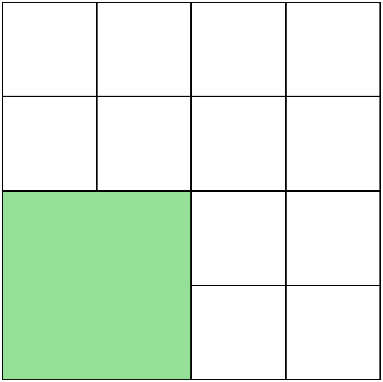

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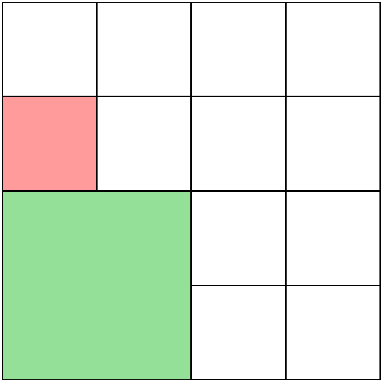

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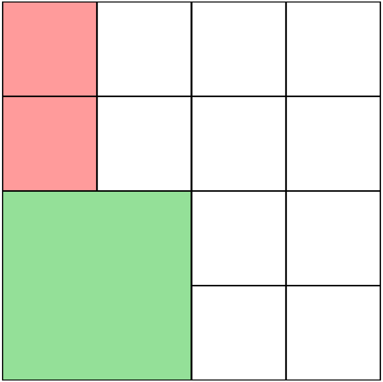

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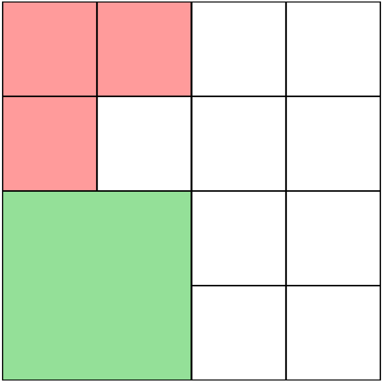

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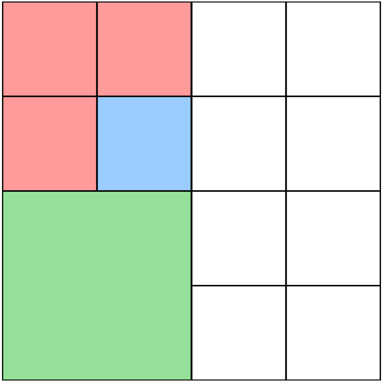

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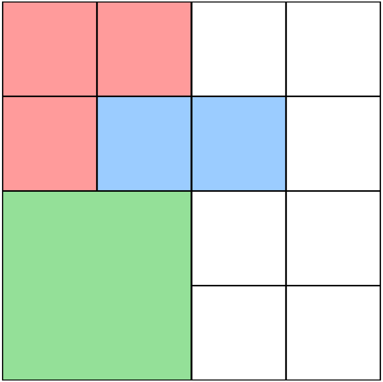

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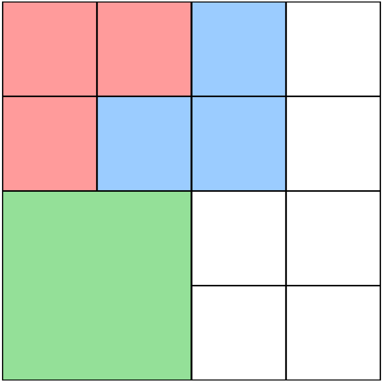

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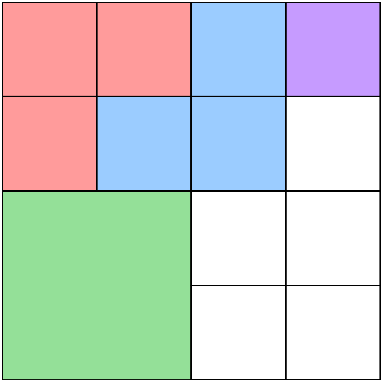

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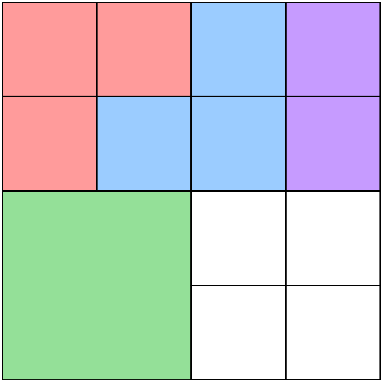

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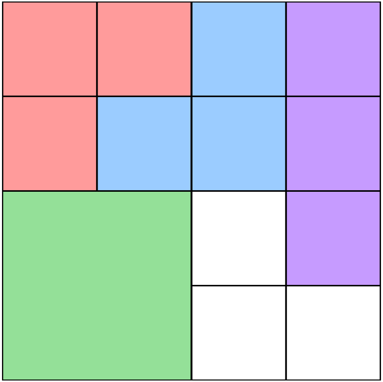

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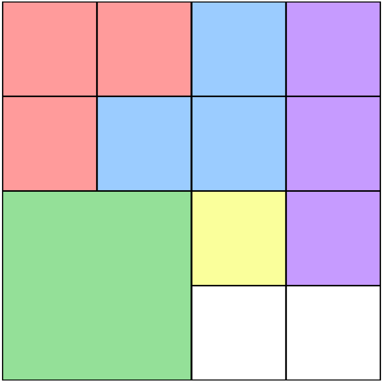

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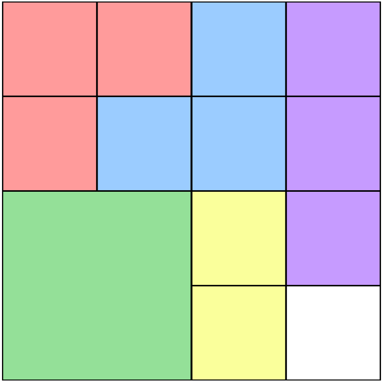

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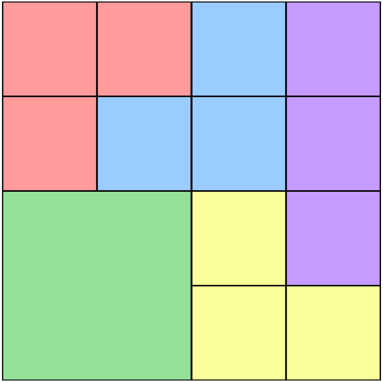

A new approach: using pseudo-Hilbert curves

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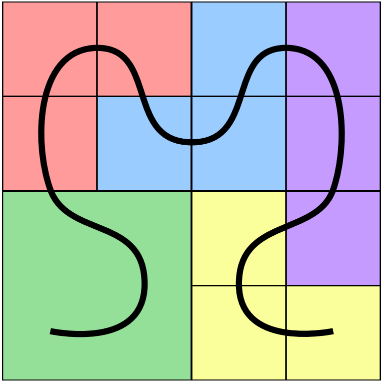
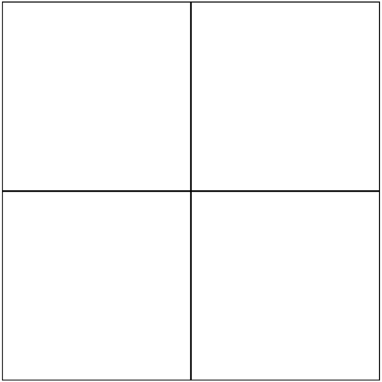


A covering game: round 1



Value used:

$$\frac{\tau}{\sqrt{N}} = \frac{1}{2}$$

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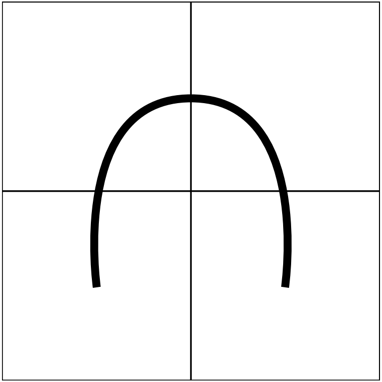

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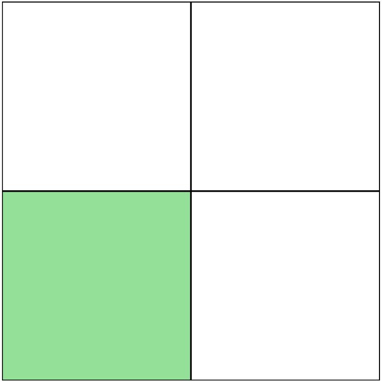

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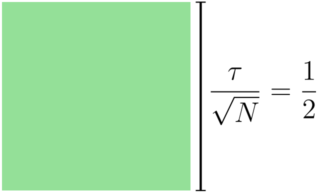

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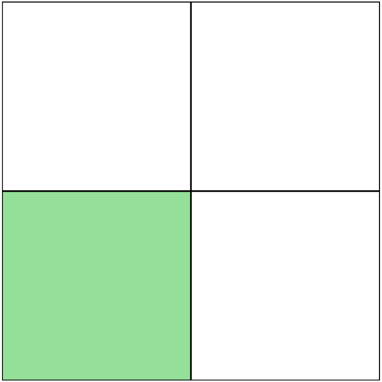

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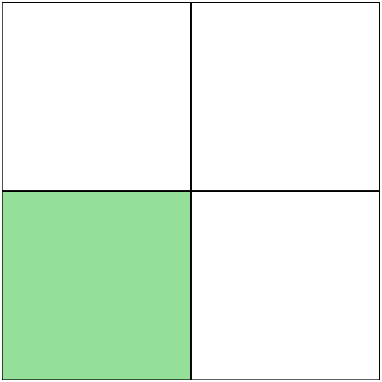
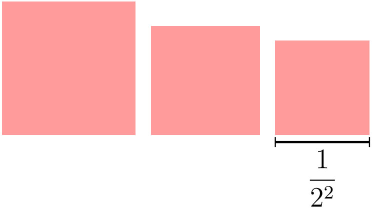
Round 1: victory!

A covering game: round 2

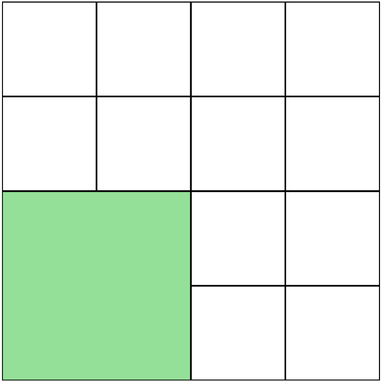


Values used:

$$\frac{\tau}{\sqrt{2N}} > \frac{\tau}{\sqrt{3N}} > \frac{\tau}{\sqrt{4N}} = \frac{1}{2^2}$$



A covering game: round 2



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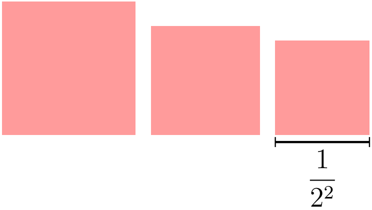

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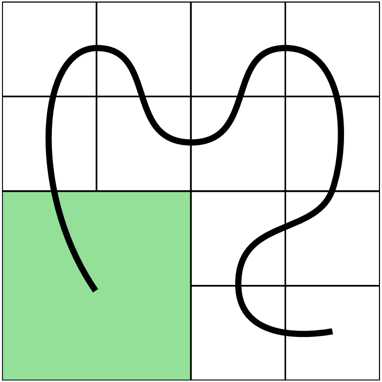

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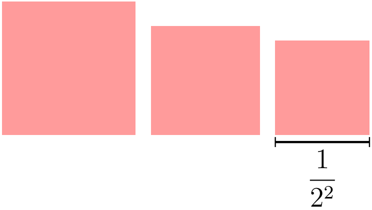
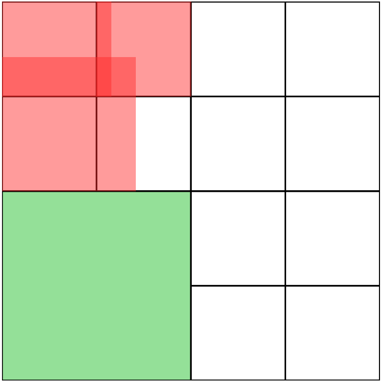
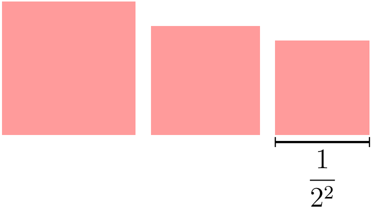


A covering game: round 2



Values used:

$$\frac{\tau}{\sqrt{2N}} > \frac{\tau}{\sqrt{3N}} > \frac{\tau}{\sqrt{4N}} = \frac{1}{2^2}$$



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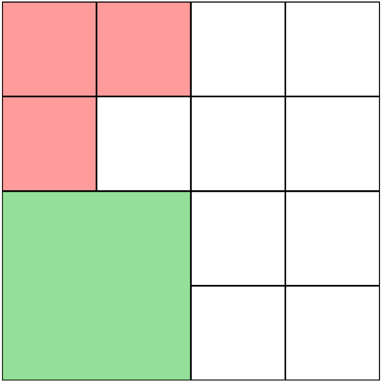

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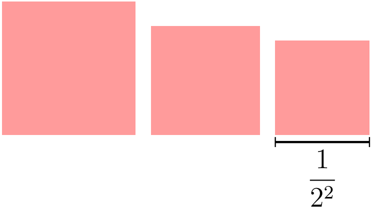

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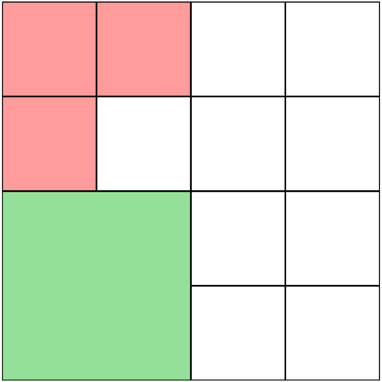

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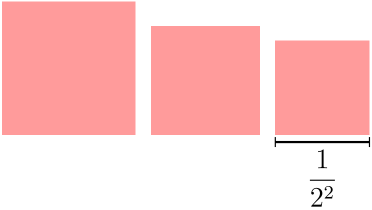
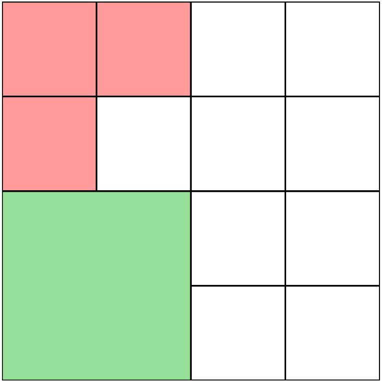


Round 2: victory!

A covering game: round 3



Values used:

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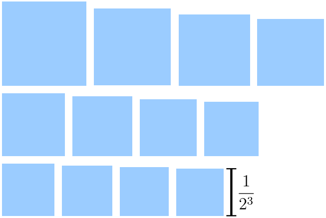
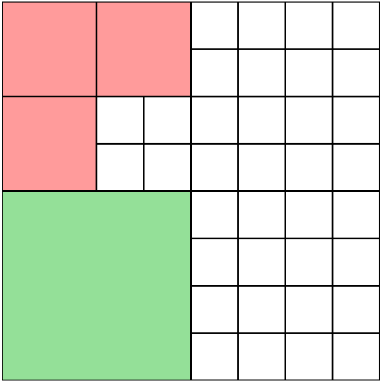


A covering game: round 3



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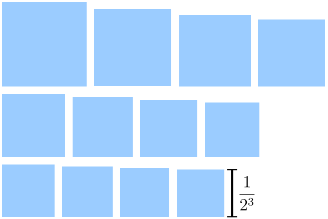

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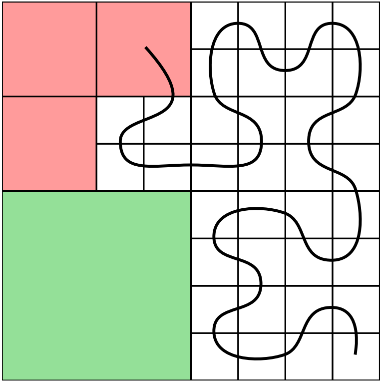

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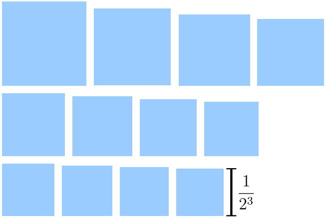

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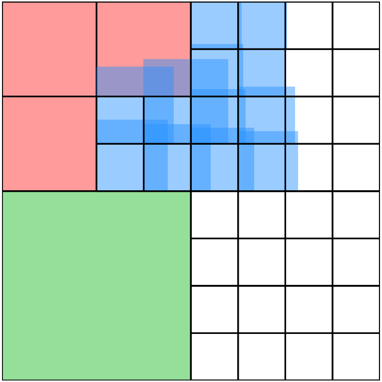

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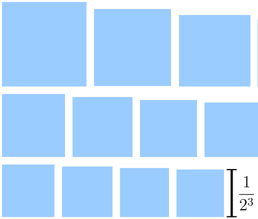

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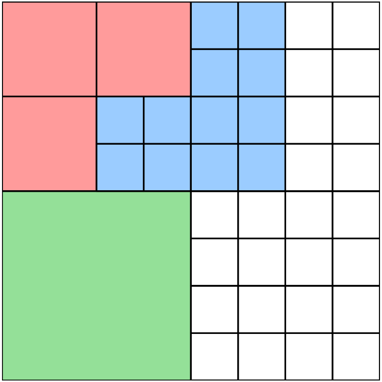

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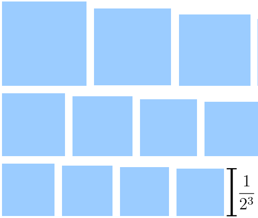
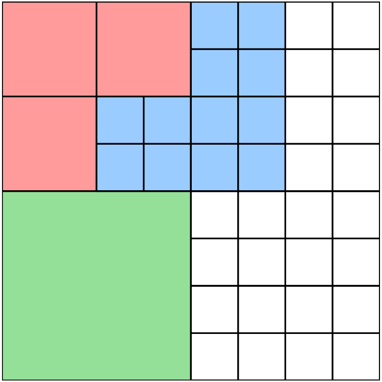


A covering game: round 3



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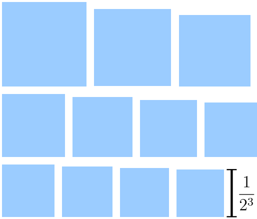
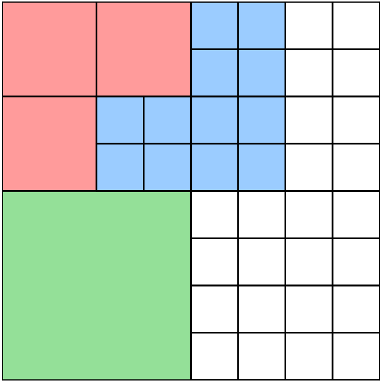


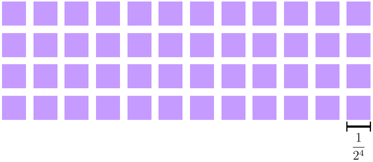
Round 3: victory!

A covering game: round 4

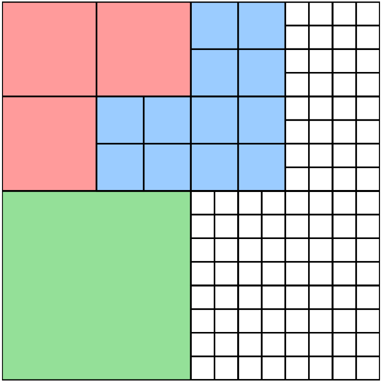


Values used:

$$\underbrace{\frac{\tau}{\sqrt{17N}} > \cdots > \frac{\tau}{\sqrt{64N}}}_{\text{to make } 48=3\times 4^2 \text{ pieces}} = \frac{1}{2^4}$$



A covering game: round 4



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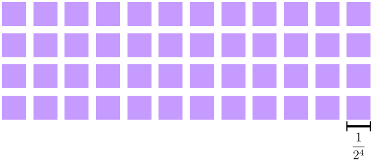
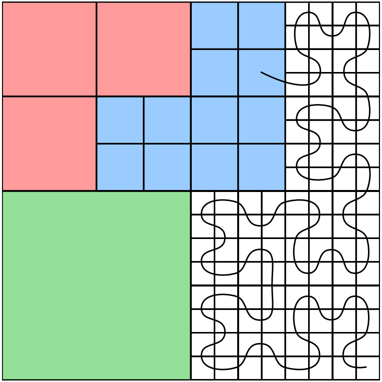


A covering game: round 4



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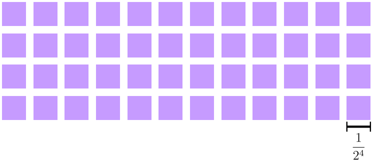
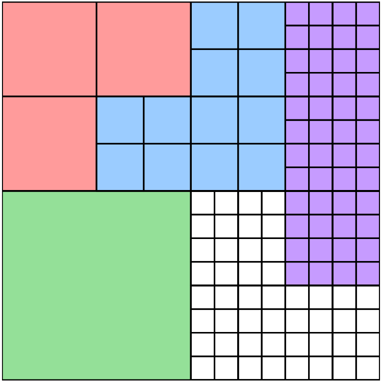


A covering game: round 4



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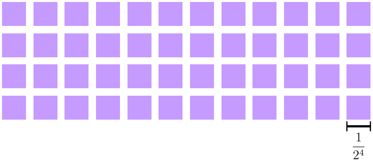

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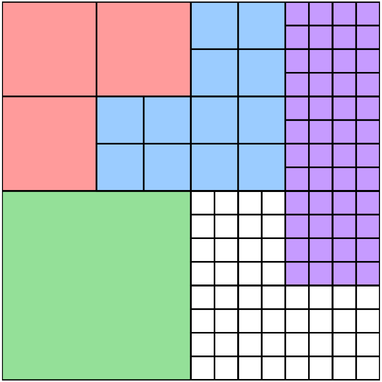

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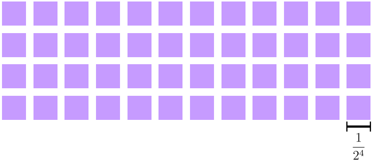
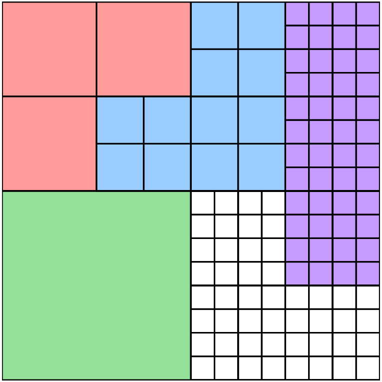


Round 4: victory!

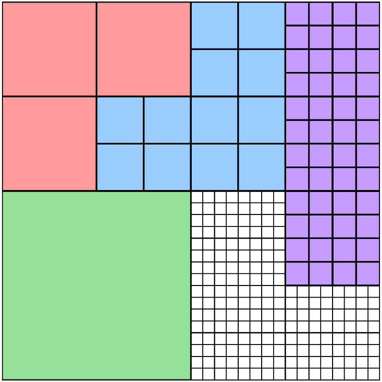
A covering game: round 5



Values used:

$$\underbrace{\frac{\tau}{\sqrt{65N}} > \cdots > \frac{\tau}{\sqrt{256N}}}_{\text{to make } 192=3\times 4^3 \text{ pieces}} = \frac{1}{2^5}$$

A covering game: round 5



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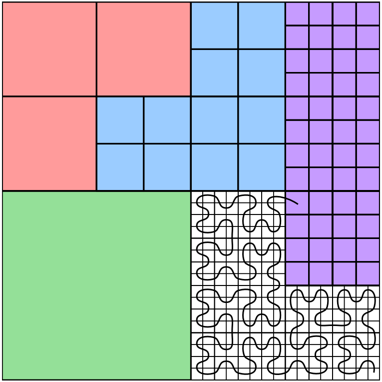
A covering game: round 5



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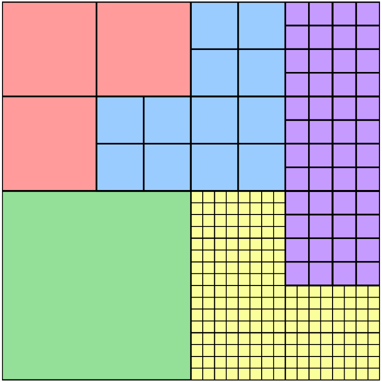

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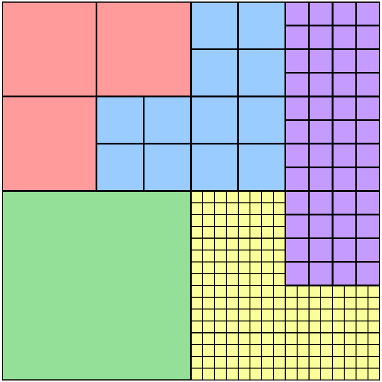

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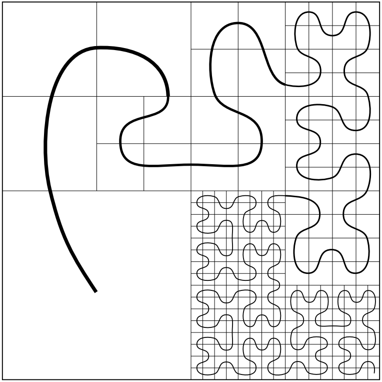
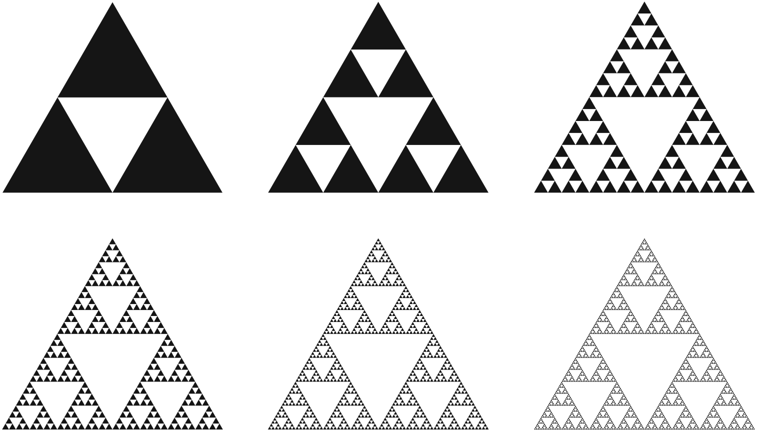
Round 5: victory. The game is over.

How the enumeration of our covering looks like



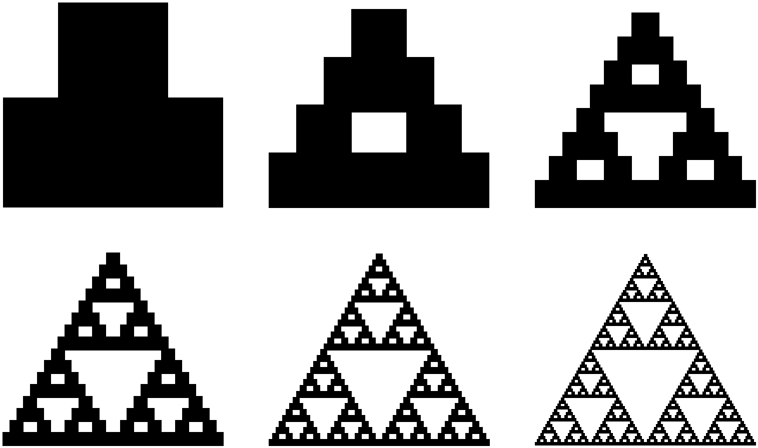
Ordering of $\Lambda_1, \Lambda_2, \ldots, \Lambda_{256}$ such that $K \subset \bigcup_{i=1}^{256} \Lambda_i$

Sierpiński gasket



Obtaining the Sierpiński gasket through triangles

Sierpiński gasket



Obtaining the Sierpiński gasket through squares

Dimension of the gasket

Let Γ be a unitary Sierpiński gasket inside $[0,+\infty)^2$. The similarity dimension (= Hausdorff dimension) of the gasket is $\dim_{\mathcal{H}} = \frac{\log 3}{\log 2}$. Up to now, we knew that the weights of the form

$$w_1(x)\cdots w_n(x) = \exp(xn^{\alpha})$$

induce a bi-dimensional family $\left(B_{w(x)} \times B_{w(y)}\right)_{(x,y)\in\Gamma}$ with a common hypercyclic vector whenever $\alpha < \frac{1}{\dim_{\mathcal{H}}\Gamma} = \frac{\log 2}{\log 3}$.

Applying the new approach to the gasket

The limit case we aim to address is $\alpha = \frac{1}{\frac{\log 3}{\log 2}} = \frac{\log 2}{\log 3}$.

Once again, we are given values $\tau > 0$, $N \in \mathbb{N}$ and we want to construct a covering by squares of sides

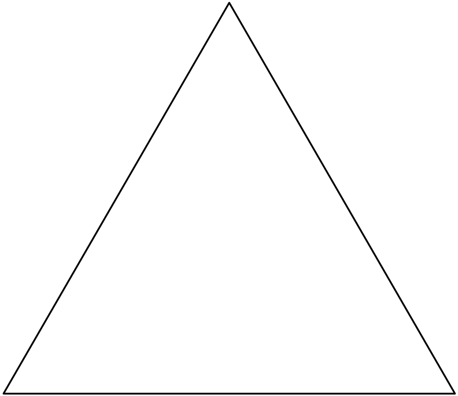
$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}}, \frac{\tau}{(2N)^{\frac{\log 2}{\log 3}}}, \frac{\tau}{(3N)^{\frac{\log 2}{\log 3}}}, \ldots$$

to make pieces for our covering.

The covering game of the gasket: grouping method

Again, let us consider the simple case

$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}} = \frac{1}{2}.$$



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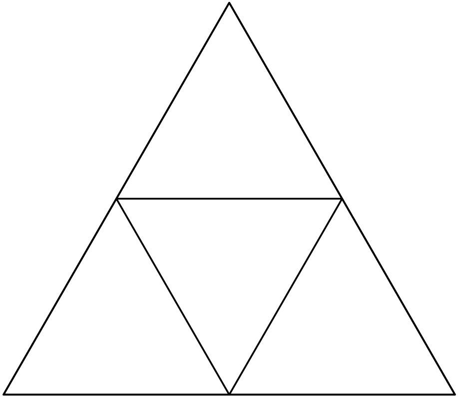

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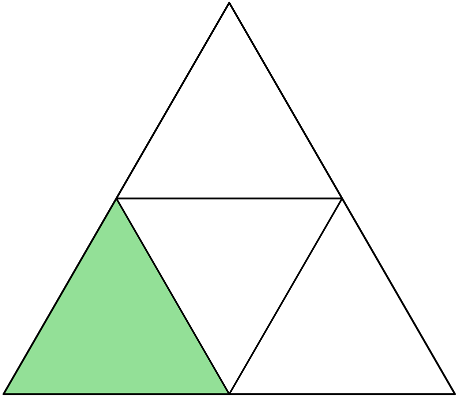

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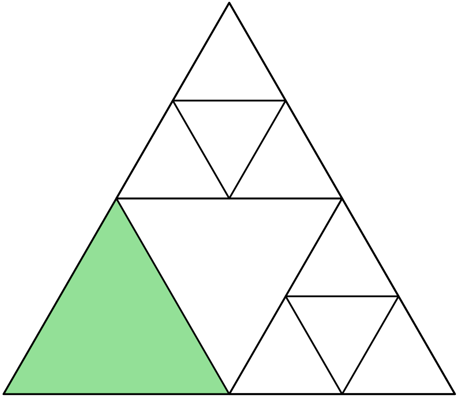

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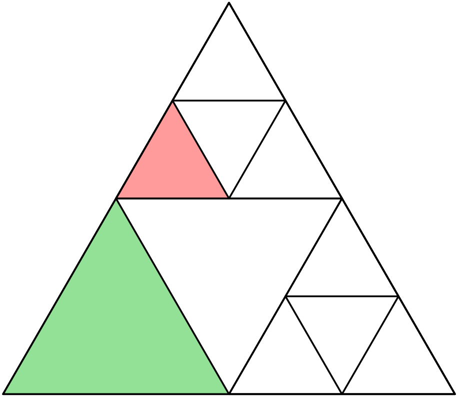

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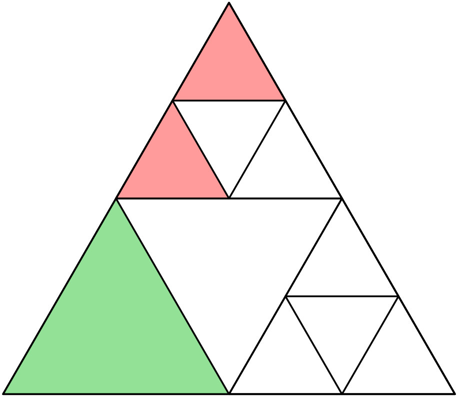

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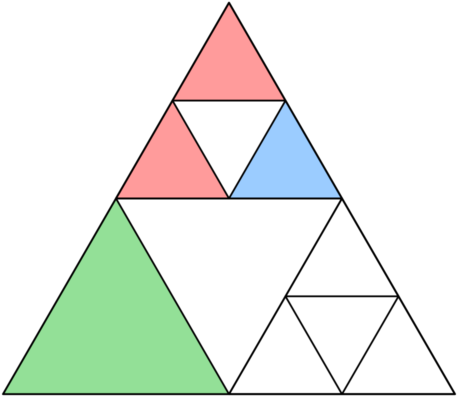

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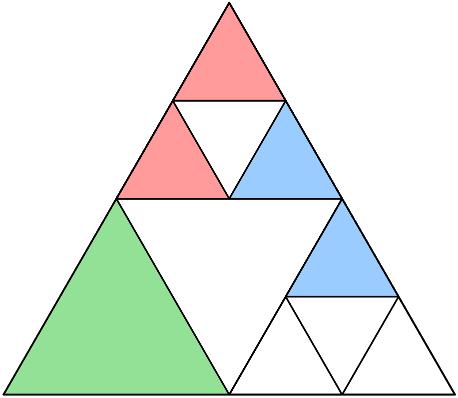

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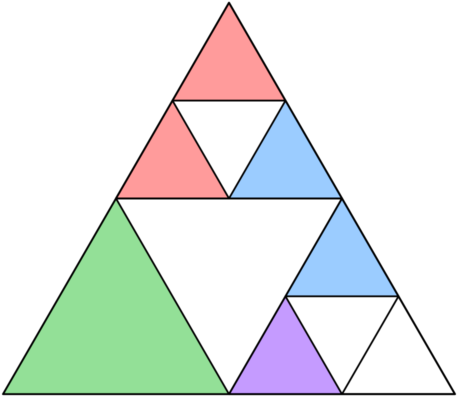

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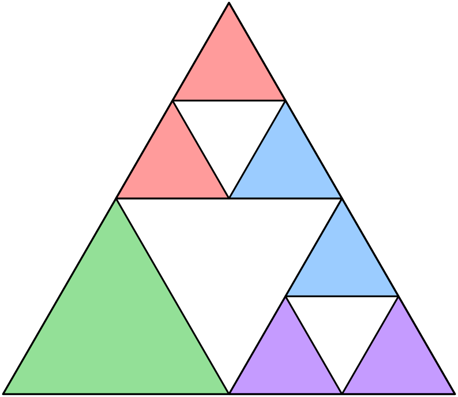

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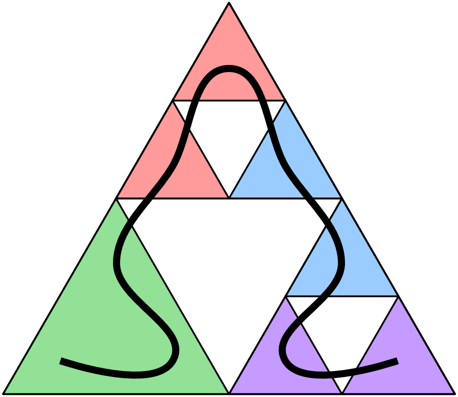
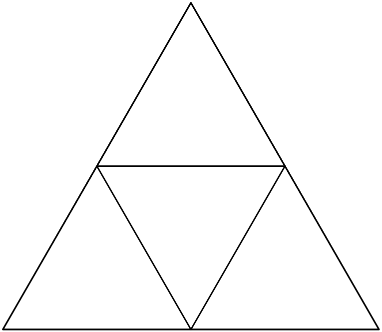


The covering game of the gasket: round 1



Value used:

$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}} = \frac{1}{2}$$

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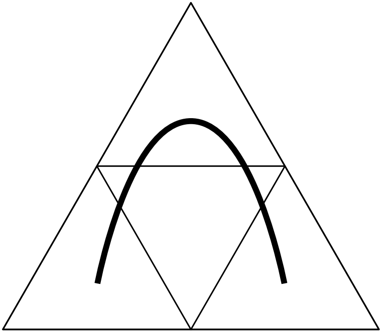

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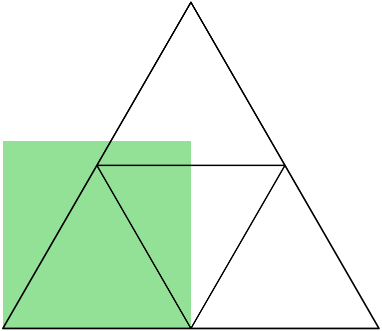

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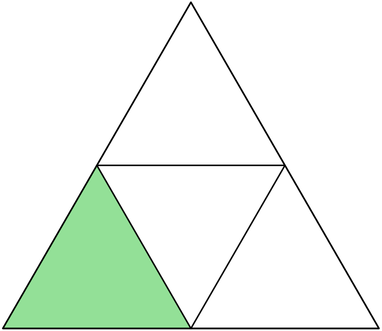

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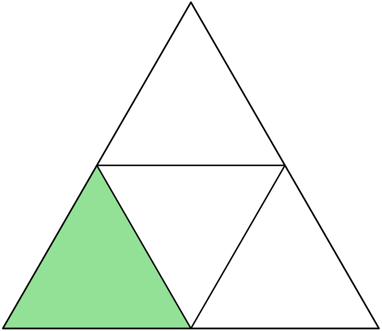
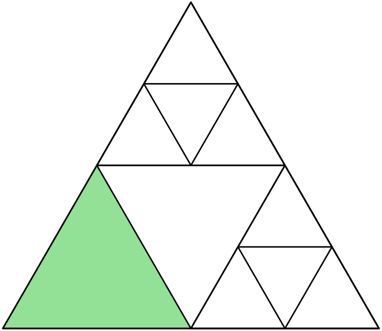


$$\frac{\tau}{N^{\frac{\log 2}{\log 3}}} = \frac{1}{2}$$

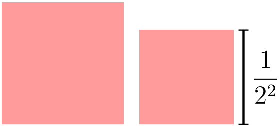
Round 1: victory!

The covering game of the gasket: round 2

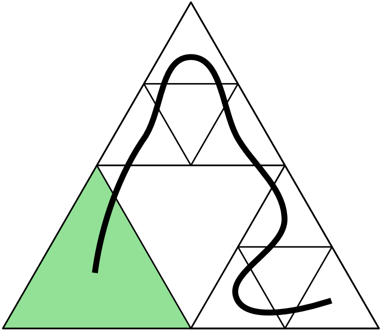


Values used:

$$\frac{\tau}{(2N)^{\frac{\log 2}{\log 3}}} > \frac{\tau}{(3N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^2}$$



The covering game of the gasket: round 2



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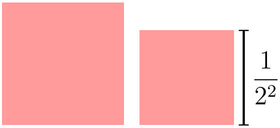
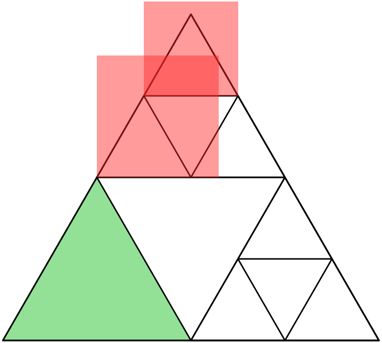


A covering game: round 2



Values used:

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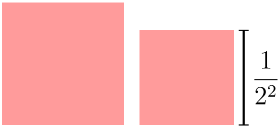
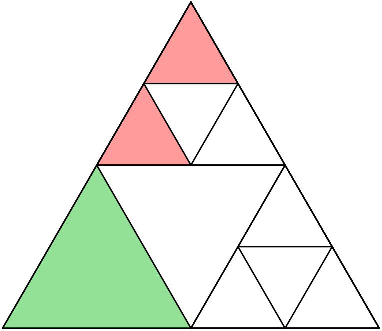


A covering game: round 2



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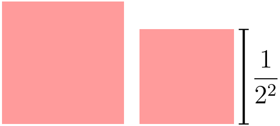
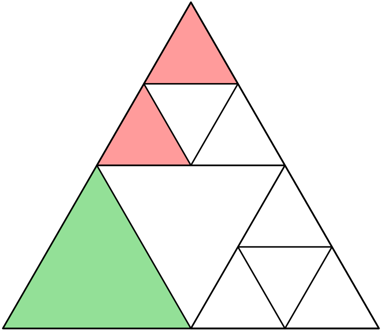


A covering game: round 2



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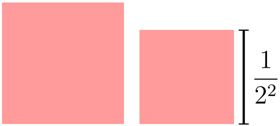
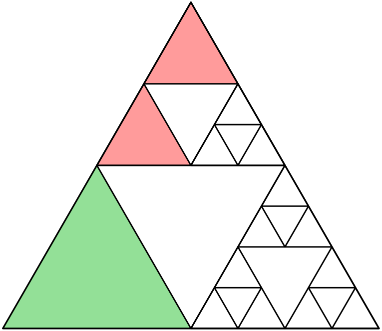


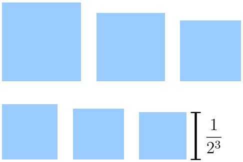
Round 2: victory!

The covering game of the gasket: round 3

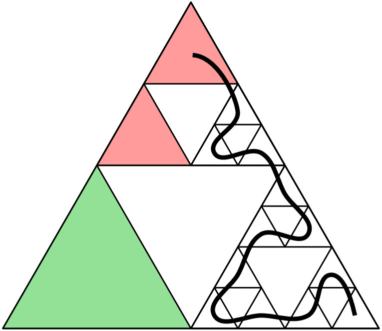


Values used:

$$\frac{\tau}{(4N)^{\frac{\log 2}{\log 3}}} > \cdots > \frac{\tau}{(9N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^3}$$



The covering game of the gasket: round 3



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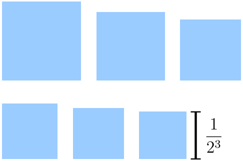
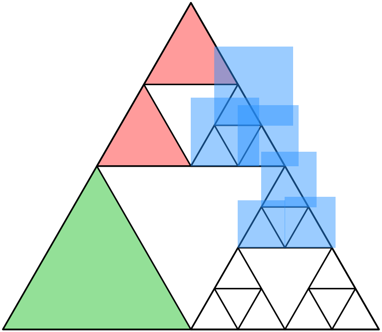


A covering game: round 3



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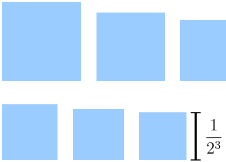

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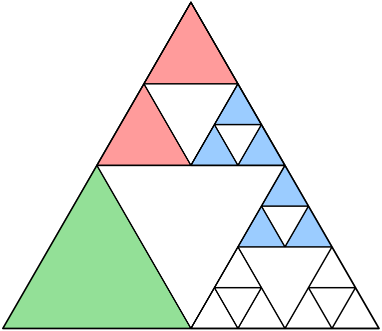

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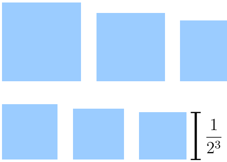
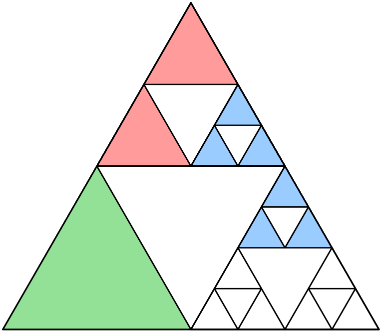


A covering game: round 3



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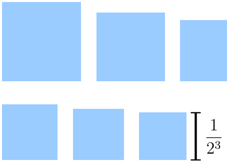
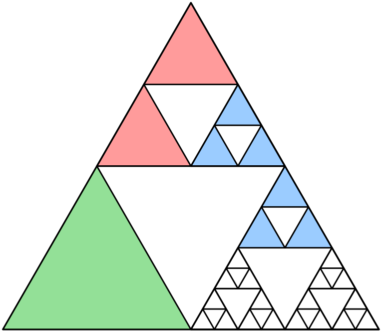


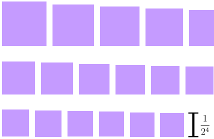
Round 3: victory!

The covering game of the gasket: round 4

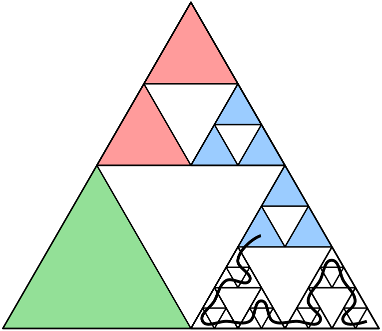


Values used:

$$\frac{\tau}{(10N)^{\frac{\log 2}{\log 3}}} > \cdots > \frac{\tau}{(27N)^{\frac{\log 2}{\log 3}}} = \frac{1}{2^3}$$



The covering game of the gasket: round 4



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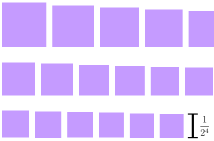
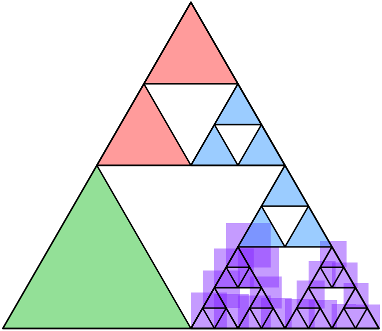


A covering game: round 4



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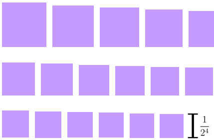
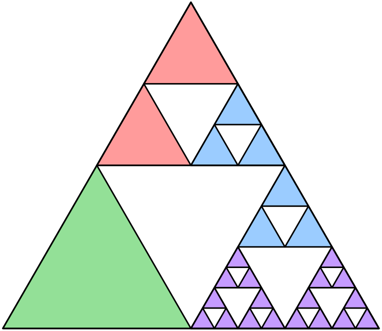


A covering game: round 4



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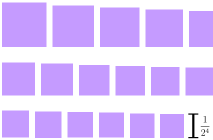
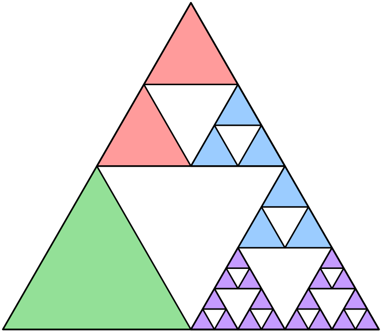


A covering game: round 4



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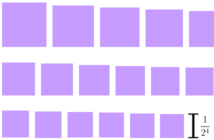
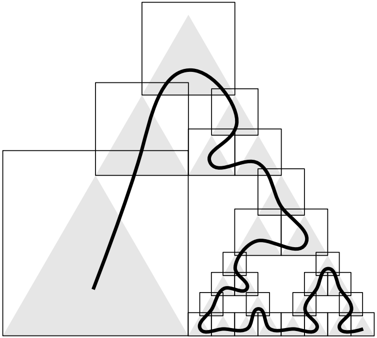


Round 4: victory! The game is over.

How the enumeration of our covering looks like



Ordering of $\Lambda_1, \Lambda_2, \ldots, \Lambda_{27}$ such that $\Gamma \subset \bigcup_{i=1}^{27} \Lambda_i$

Koch snowflake

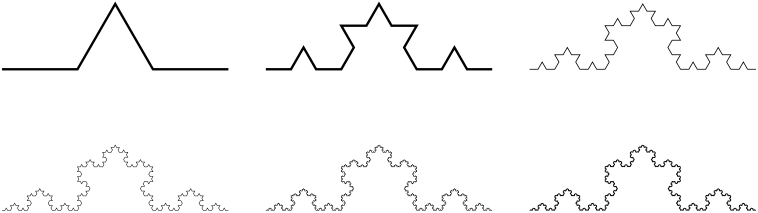


Figure : The von Koch curve forms the well known Koch snowflake. It has 4 similarities and contraction ration $\frac{1}{3}$. Its has Hausdorff dimension $\frac{\log(4)}{\log(3)}$.

We can apply our method by making groups of 3 patches of parts.

Minkowski sausage

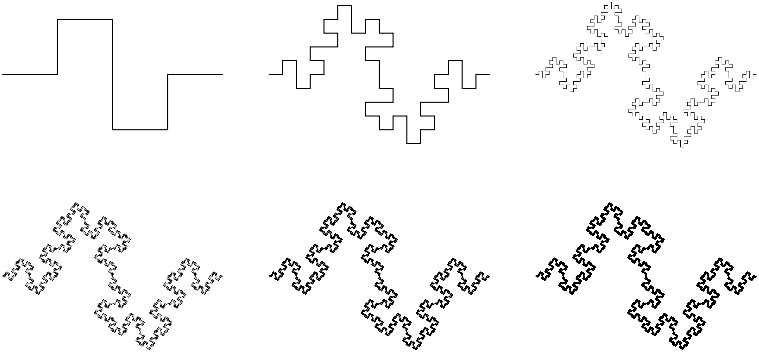


Figure : The Minkowski sasusage has 8 similarities with $\frac{1}{4}$ contraction ration. Its Hausdorff dimension is 1.5.

We can apply our method by making groups of 7 patches of parts.

A new concept of dimension?

Definition. Let $d \geq 1$ and $\Lambda \subset \mathbb{R}^d$ be compact. We say that Λ has *homogeneously ordered box dimension at most* $\gamma \in (0, d]$ if there exist $r \in [\![2, +\infty[\![$, $\rho > 0$ and, for all $m \geq 1$, a compact covering $(\Lambda_{\mathbf{i}})_{\mathbf{i} \in I_r^m}$ of Λ satisfying the following.

(i) For all $\mathbf{i} \in I_r^m$,

$$\operatorname{diam}(\Lambda_{\mathbf{i}}) \leq \rho \left(\frac{1}{r^{1/\gamma}} \right)^m .$$

(ii) For all $\mathbf{i} = (i_1, ..., i_m, i_{m+1}) \in I_r^{m+1}$,

$$\Lambda_{i_1,\ldots,i_{m+1}} \subset \Lambda_{i_1,\ldots,i_m}.$$

(iii) For all $\mathbf{i} = (i_1, ..., i_m) \in I_r^m$ and all $j \in \{2, ..., r\}$,

$$\Lambda_{\mathbf{i},j-1,r} \cap \Lambda_{\mathbf{i},j,1} \neq \varnothing.$$

The *homogeneously ordered box dimension of* Λ (HBD$^\circ$ for short) is the smallest number γ such that Λ has homogeneously ordered box dimension at most γ. We will denote this number by $\dim_{HB}^\circ(\Lambda)$.

A full generalization of the CS-criterion

Theorem (Generalized Costakis-Sambarino Criterion)

Let $d \in \mathbb{N}$ *and* $\gamma > 0$ *and let* Λ *be a compact subset of* $\mathbb{R}^d$ *having homogeneously ordered box dimension at most* γ*. Assume that there is* $D > 0$ *such that, for all* $u \in \mathcal{D}$ *and all compact* $K \subset \Lambda$*, the following properties hold true, where all parameters* λ, μ *belong to* K*.*

(1) *There exist* $\kappa \in \mathbb{N}$ *and a sequence of positive numbers* $(c_k)_{k\in\mathbb{N}}$ *such that* $\sum_{k=\kappa}^{\infty} c_k < \infty$ *and, whenever* $\|\lambda - \mu\|_\infty \leq D\frac{k^{1/\gamma}}{(n+k)^{1/\gamma}}$*, with* $n, k \in \mathbb{N}_0$ *and* $k \geq \kappa$*, we have*

$$\left\|T_\lambda^{n+k} S_\mu^n u\right\| \leq c_k \qquad \text{and} \qquad \left\|T_\lambda^n S_\mu^{n+k} u\right\| \leq c_k.$$

(2) *Given* $\eta > 0$*, one can find* $\tau > 0$ *such that, for all* $n \in \mathbb{N}$*,*

$$\|\lambda - \mu\|_\infty \leq \frac{\tau}{n^{1/\gamma}} \implies \left\|T_\lambda^n S_\mu^n u - u\right\| < \eta.$$

Then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ *is a* G_δ*-dense subset of* X*.*

Applications

Let $d \geq 1$ and $\Gamma \subset \mathbb{R}^d$ compact. We write $\lambda = (x_1, \ldots, x_d)$ for any $\lambda \in \mathbb{R}^d$. Consider $\alpha \in (0, 1/d]$ and a family $\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big)_{\lambda \in \Gamma}$ induced by weights $\big(w(x)\big)_{x \in \mathbb{R}}$ satisfying

$$w_1(x) \cdots w_d(x) \approx \exp(x n^{\alpha}).$$

Theorem

If Γ has HBO$^{\circ}$ at most $\gamma \leq d$ and $0 < \alpha \leq 1/\gamma$, then

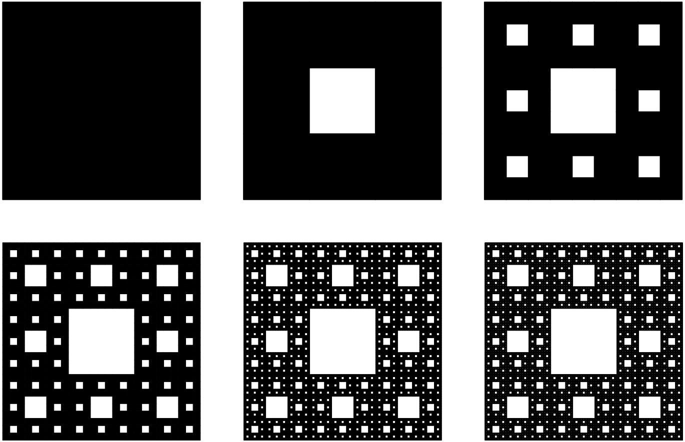
$$\bigcap_{\lambda \in \Gamma} HC\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big) \neq \varnothing.$$

Corollary

If Γ is a β-Hölder curve and $0 < \alpha \leq \beta$, then

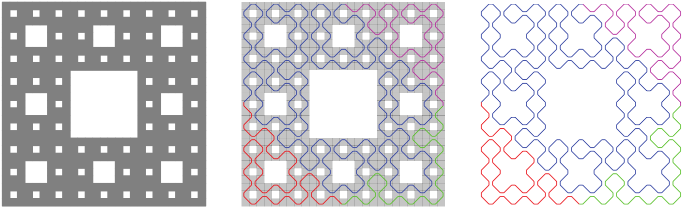
$$\bigcap_{\lambda \in \Gamma} HC\big(B_{w(x_1)} \times \overset{d}{\cdots} \times B_{w(x_d)}\big) \neq \varnothing.$$

IFS, GIFS and optimal parametrizations



Construction of the Sierpiński carpet.

IFS, GIFS and optimal parametrizations



Ordering the carpet (Rao and Zhang, 2016).

Some open questions

- $\sum \frac{1}{F(n)^2} = +\infty \implies \bigcap_{x,y>0} HC(B_{w(x)} \times B_{w(y)}) \neq \varnothing$?

- For $\mathcal{C} \subset \mathbb{R}^2_+$ the Cantor dust with dissection ratio $1/4$,

$$\bigcap_{(x,y)\in\mathcal{C}} e^x B \times e^y B \neq \varnothing?$$

- Can we, more generally, generalize our result for totally disconnected fractals?

- Can we adapt optimally our technique to self-similar fractals with non-uniform contraction ratio?

- For $\mathcal{T} \subset \mathbb{R}^2_+$ the 1-dimensional Takagi curve,

$$\bigcap_{(x,y)\in\mathcal{T}} e^x B \times e^y B \neq \varnothing?$$

References

[1] E. Abakumov and J. Gordon. **Common hypercyclic vectors for multiples of backward shift.** *Journal of Functional Analysis* 200.2 (2003): 494-504.

[2] G. Costakis and M. Sambarino. **Genericity of wild holomorphic functions and common hypercyclic vectors.** *Advances in Mathematics* 182.2 (2004): 278-306.

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[4] F. Bayart, F. Costa Jr. and Q. Menet. **Common hypercyclic vectors and dimension of the parameter set.** *Indiana University Mathematics Journal* 71.4 (2022): 1763-1795.

[5] A. Peris. **Common hypercyclic vectors for backward shifts**, *Operator Theory Seminar*, Michigan State University (2000/01).

[6] H. Rao and S.-Q. Zhang. **Space-filling curves of self-similar sets (I): iterated function systems with order structures.** *Nonlinearity* 29.7 (2016): 2112.