Analyse 2 - TD 4 (2023)

Exo. 1

▷ $$\frac{\frac{1}{n}}{1} = \frac{1}{n} \longrightarrow 0 \Rightarrow \frac{1}{n} = o(1)$$

▷ $$\frac{\frac{(\ln(n))^{10}}{n^3}}{\frac{1}{n^2}} = \frac{(\ln(n))^{10}}{n^3} \cdot n^2 = \frac{(\ln(n))^{10}}{n} \longrightarrow 0 \Rightarrow \frac{(\ln(n))^{10}}{n^3} = o\left(\frac{1}{n^2}\right)$$

▷ $$\frac{\frac{2n+3}{n^2-5}}{\frac{1}{n}} = \frac{2n+3}{n^2-5} \cdot n = \frac{2n^2+3n}{n^2-5} = \frac{2+\frac{3}{n}}{1-\frac{5}{n}} \longrightarrow 2 \text{ converge}$$

$$\Rightarrow \left(\frac{\frac{2n+3}{n^5-5}}{\frac{1}{n}}\right)_n \text{ bornée} \Rightarrow \frac{2n+3}{n^5-5} = O\left(\frac{1}{n}\right).$$

▷ $$\frac{\frac{\cos\left(\frac{n\pi}{3}\right)}{n}}{1} = \cos\left(n\frac{\pi}{3}\right) \cdot \frac{1}{n} \longrightarrow 0 \Rightarrow \frac{\cos\left(\frac{n\pi}{3}\right)}{n} = o(1)$$

$$\Rightarrow \frac{\cos\left(\frac{n\pi}{3}\right)}{n} = O(1).$$

Exo. 2 Devoir

Exo. 3

1) On a $n^6+4n^2-6 \sim n^6$, car

$$\lim_{n\to\infty} \frac{n^6+4n^2-6}{n^6} = \lim_{n\to\infty}\left(1+\frac{4}{n^4}-\frac{6}{n^6}\right) = 1,$$

et $7n^4-3n^2+n \sim 7n^4$ car

$$\lim \frac{7n^4-3n^2+n}{7n^4} = \lim\left(1-\frac{3}{7n^2}+\frac{1}{7n^3}\right) = 1.$$

Ainsi, $$\frac{n^6+4n^2-6}{7n^4-3n^2+n} \sim \frac{n^6}{7n^4} = \frac{n^2}{7}, \text{ lorsque } n \to +\infty.$$

2) On a

$$\frac{1}{\sqrt{n+1}} \longrightarrow 0 \Rightarrow \sin\left(\frac{1}{\sqrt{n+1}}\right) \sim \frac{1}{\sqrt{n+1}}.$$

En plus $\sqrt{n+1} \sim \sqrt{n}$ car

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = \sqrt{1} = 1.$$

Ainsi,

$$\sin\left(\frac{1}{\sqrt{n+1}}\right) \sim \frac{1}{\sqrt{n+1}} \sim \frac{1}{\sqrt{n}}, \text{ lorsque } n \to +\infty.$$

3) On a $n^3 - \sqrt{n^2+1} \sim n^3$, car

$$\lim_{n\to+\infty} \frac{n^3 - \sqrt{n^2+1}}{n^3} = \lim_{n\to+\infty}\left(1 - \frac{\sqrt{n^2+1}}{n^3}\right) = 1,$$

et $\ln(n) - 2n^2 \sim -2n^2$, car

$$\lim_{n\to\infty} \frac{\ln(n)-2n^2}{-2n^2} = \lim_{n\to\infty}\left(\frac{-\ln(n)}{2n^2} + 1\right) = 1.$$

Ainsi,

$$\frac{n^3 - \sqrt{n^2+1}}{\ln(n) - 2n^2} \sim \frac{n^3}{-2n^2} = -\frac{n}{2}, \text{ quand } n \to +\infty.$$

Exo. 4.1) On a $(n+1)^{1/3} - n^{1/3} = n^{1/3}\left(1+\frac{1}{n}\right)^{1/3} - n^{\frac{1}{3}} = n^{1/3}\left[\left(1+\frac{1}{n}\right)^{\frac{1}{3}} - 1\right].$

Puisque $\frac{1}{n} \to 0$, on trouve

$$(n+1)^{1/3} - n^{1/3} = n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3} - 1\right] \sim n^{1/3} \cdot \frac{1}{3} \cdot \frac{1}{n} = \frac{1}{3n^{2/3}}$$

"Ici, nous avons utilisé que $u_n \to 0 \Rightarrow (1+u_n)^{\alpha} - 1 \sim \alpha u_n$."

2) On a

$$\frac{1}{n} \longrightarrow 0 \Rightarrow \sin\left(\frac{1}{n}\right) \longrightarrow 0 \quad \text{et} \quad \underbrace{\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}}_{(1)}$$

$$\Rightarrow 1 + \sin\left(\frac{1}{n}\right) \longrightarrow 1$$

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$$\Rightarrow \ln\left(1+\sin\left(\tfrac{1}{n}\right)\right) \to 0 \quad \text{et} \quad \underbrace{\ln\left(1+\sin\left(\tfrac{1}{n}\right)\right) \sim \sin\left(\tfrac{1}{n}\right)}_{(2)}$$

$$\Rightarrow \left(1+\ln\left(1+\sin\left(\tfrac{1}{n}\right)\right)\right)^{2/3} - 1 \sim \frac{2}{3}\ln\left(1+\sin\left(\tfrac{1}{n}\right)\right)$$

$$\overset{(2)}{\sim} \frac{2}{3}\sin\left(\tfrac{1}{n}\right)$$

$$\overset{(1)}{\sim} \frac{2}{3}\cdot\frac{1}{n} = \frac{2}{3n}$$

$$\Rightarrow \left(1+\ln\left(1+\sin\left(\tfrac{1}{n}\right)\right)\right)^{2/3} - 1 \sim \frac{2}{3n}.$$

Exo. 5

1) On a

$$\frac{\pi}{2n^4} \to 0^+ \Rightarrow \sin\left(\frac{\pi}{2n^4}\right) \to 0^+ \quad \text{et} \quad \underbrace{\sin\left(\frac{\pi}{2n^4}\right) \sim \frac{\pi}{2n^4}}_{(*)}$$

$$\Rightarrow \sqrt{\sin\left(\frac{\pi}{2n^4}\right)} \overset{(*)}{\sim} \sqrt{\frac{\pi}{2n^4}} = \frac{1}{n^2}\sqrt{\frac{\pi}{2}} \quad (\heartsuit$$

$$\Rightarrow \underbrace{(n^2+n)}_{\sim n^2}\sqrt{\sin\left(\frac{\pi}{2n^4}\right)} \overset{\heartsuit}{\sim} n^2 \cdot \frac{1}{n^2}\sqrt{\frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}$$

Ainsi,

$$\lim_{n\to\infty} (n^2+n)\sqrt{\sin\frac{\pi}{2n^4}} = \sqrt{\frac{\pi}{2}}.$$

2) On a

$$\frac{1}{n} \to 0 \Rightarrow \cos\frac{1}{n} \to 1 \Rightarrow \sqrt{\cos\frac{1}{n}} - 1 \to 0,$$

donc

$$\tan\left(\sqrt{\cos\frac{1}{n}} - 1\right) \sim \sqrt{\cos\frac{1}{n}} - 1$$

Or,

$$\cos\frac{1}{n} \to 1 \Rightarrow \cos\frac{1}{n} - 1 \to 0$$

Ainsi,

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$$\sqrt{\cos\left(\frac{1}{n}\right)}-1=\sqrt{1+\underbrace{\left(\cos\left(\frac{1}{n}\right)-1\right)}_{\to 0}}-1\sim\frac{1}{2}\left(\cos\left(\frac{1}{n}\right)-1\right)$$

En plus,

$$\frac{1}{n}\to 0 \Rightarrow \cos\left(\frac{1}{n}\right)-1=-\left(1-\cos\left(\frac{1}{n}\right)\right)\sim-\frac{\left(\frac{1}{n}\right)^2}{2}=-\frac{1}{2n^2}.$$

Par conséquent,

$$n^2\tan\left(\sqrt{\cos\frac{1}{n}}-1\right)\sim n^2\left(\sqrt{\cos\frac{1}{n}}-1\right)$$

$$\sim n^2\frac{1}{2}\left(\cos\left(\frac{1}{n}\right)-1\right)$$

$$\sim n^2\frac{1}{2}\cdot\frac{-1}{2n^2}=\frac{-1}{4}.$$

Ainsi,

$$\lim_{n\to\infty} n^2\tan\left(\sqrt{\cos\frac{1}{n}}-1\right)=\frac{-1}{4}.$$

<u>Exo. 6</u> 3) On écrit

$$\left(\cos\frac{1}{n}\right)^{n^2}=\exp\left(n^2\ln\left(\cos\frac{1}{n}\right)\right).$$

On note que $\frac{1}{n}\to 0^+ \Rightarrow \cos\frac{1}{n}\to 1$. Ainsi,

$$\ln\left(\cos\frac{1}{n}\right)=\ln\left(1+\underbrace{\left(\cos\frac{1}{n}-1\right)}_{\to 0}\right)\sim\cos\frac{1}{n}-1.$$

Donc

$$n^2\ln\left(\cos\frac{1}{n}\right)\sim n^2\left(\cos\frac{1}{n}-1\right)=-n^2\left(1-\cos\frac{1}{n}\right)\sim-n^2\frac{\left(\frac{1}{n}\right)^2}{2}$$

$$\Rightarrow n^2\ln\left(\cos\left(\frac{1}{n}\right)\right)\sim\frac{-n^2}{2n^2}=-\frac{1}{2}$$

$$\Rightarrow \exp\left(\ln\left(\cos\left(\frac{1}{n}\right)\right)\right.\sim\exp\left(-\frac{1}{2}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\cos\left(\frac{1}{n}\right)\right)^{n^2}=\exp\left(-\frac{1}{2}\right).$$

5 / 4) On sait que $\frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4} \to 0$, donc

$$\sin\left(\frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4}\right) \sim \frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4}.$$

En plus, $\frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4} \sim \frac{1}{n^2}$, car

$$\frac{\frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4}}{\frac{1}{n^2}} = 1+\frac{2n^2}{n^3}-\frac{n^2}{n^4} = 1+\frac{2}{n}-\frac{1}{n^2} \to 1.$$

Ainsi,

$$\sin\left(\frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4}\right) \sim \frac{1}{n^2}.$$

D'autre part, puisque $\frac{4}{n}-\frac{3}{n^2} \to 0$, on a

$$\exp\left(\frac{4}{n}-\frac{3}{n^2}\right)-1 \sim \frac{4}{n}-\frac{3}{n^2} \overset{*}{\sim} \frac{4}{n}$$ *(devoir)

$$\Rightarrow \sqrt{\exp\left(\frac{4}{n}-\frac{3}{n^2}\right)-1} \sim \sqrt{\frac{4}{n}} = \frac{2}{\sqrt{n}}.$$

Par conséquent,

$$\frac{\sin\left(\frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4}\right)}{\sqrt{\exp\left(\frac{4}{n}-\frac{3}{n^2}\right)-1}} \sim \frac{\frac{1}{n^2}}{\frac{2}{\sqrt{n}}} = \frac{\sqrt{n}}{2n^2} = \frac{1}{2n^{3/2}} \to 0,$$

d'où

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}+\frac{2}{n^3}-\frac{1}{n^4}\right)}{\sqrt{\exp\left(\frac{4}{n}-\frac{3}{n^2}\right)-1}} = 0.$$